Retrograde motion: The Derivation of Formula

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This refers to the paper entitled "Retrograde motion as described in *Brahmatulyaudāharaṇam* of Viśvanātha" BS Subha and BS Shylaja published in *IJHS* 55.1 (2020): 40–48. The explanation for retrograde motion of a planet is based on the concept that the planet appears stationary as seen from the earth owing to the relative motion. The standard formula used for the derivation of stationary points in the orbit of a planet involves the condition that the sum of velocities be zero. Here we provide the standard derivation from textbooks and an alternative method of arriving at the same result. Further we compare it with the method provided in *Karaṇakutūhala* of Bhāskarācārya and show that the same result has been arrived at by Bhāskarācārya as cited in the paper on the examples from *Brahmatulyaudāharaṇam*.

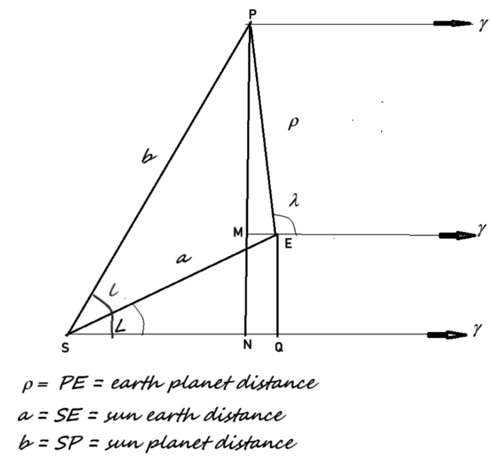

Figure 1 The geometry of the problem.

1 Formulation of the problem

Let us consider the procedure as taught today from conventional text books. The purpose is to find the condition for the difference in the longitudes of the sun and the planet is zero. Figure 1 depicts the positions of the earth, the sun and the planet. The longitudes are measured with the reference direction of γ, the First Point of Aries. The earth-sun distance is indicated by a, planet-sun distance is indicated by b, the earth-planet distance by ρ. The longitudes are also marked. For the derivation we draw perpendiculars PN and EQ on to Sγ and mark the point M so that EM is parallel and hence equal to NQ.

We form a set of equations from the basic trigonometric relation from the triangle

$PN = PM + MN$ can be written as

$$b\ sinl = \rho\ sin\ PEM + EQ = \rho\ sin\lambda + a\ Sin\ L \qquad (1)$$

In this equation ρ l, λ and L are all varying with time. Therefore, by differentiating we get

$$b\ cos\ l\ \frac{dl}{dt} = \rho\ cos\ \lambda \frac{d\lambda}{dt} + sin\ \lambda \frac{d\rho}{dt} + a\ cos\ L \frac{dL}{dt}$$

We multiply this by $\rho\ cos\ \lambda$ to get

$$b\ \rho\ cos\ \lambda\ cos\ l \frac{dl}{dt} = \rho^2 cos^2\ \lambda \frac{d\lambda}{dt} + \rho\ cos\ \lambda\ sin\ \lambda \frac{d\rho}{dt} + \rho\ cos\ \lambda\ a\ cos\ L \frac{dL}{dt}$$

Rearranging

$$\rho^2 cos^2 \lambda \frac{d\lambda}{dt} = b\ \rho\ cos\ \lambda\ cos\ l \frac{dl}{dt} - \rho\ cos\ \lambda\ sin\ \lambda \frac{d\rho}{dt} - \rho\ cos\ \lambda\ a\ cos\ L \frac{dL}{dt} \qquad (2)$$

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Let us multiply (1) by *sinl*

$$b\,sin^2 l = \rho\,sin\,l\,sin\,\lambda + a\,sin\,l\,sin\,L \qquad (3)$$

Let us multiply (1) by *sinL*

$$b\,sin\,L\,sin\,l = \rho\,sin\,L\,sin\,\lambda + a\,sin^2 L \qquad (4)$$

Let us now get three more equations with the other identity derived from

$SQ = SN + NQ$ which can be written as

$$a\,cos\,L = b\,cos\,l + ME = b\,cos\,l + \rho\,cos\,PEM = b\,cos\,l - \rho\,cos\,\lambda \qquad (5)$$

Differentiating this equation for the variables, we get

$$-a\,sin\,L\frac{dL}{dt} = -b\,sin\,l\frac{dl}{dt} - \rho\,sin\,\lambda\frac{d\lambda}{dt} - cos\,\lambda\frac{d\rho}{dt}$$

Multiplying this $\rho sin\lambda$ we get

$$-a\,\rho\,sin\,\lambda\,sin\,L\frac{dL}{dt} = -b\,\rho\,sin\,\lambda\,sin\,l\,\frac{dl}{dt} - \rho\,sin\,\lambda\,cos\,\lambda\frac{d\rho}{dt} - \rho^2 sin^2\lambda\frac{d\lambda}{dt}$$

Or

$$\rho^2 sin^2\lambda\frac{d\lambda}{dt} = a\,\rho\,sin\,\lambda\,sin\,L\,\frac{dL}{dt} - b\,\rho\,sin\,\lambda\,sin\,l\,\frac{dl}{dt} - \rho\,sin\,\lambda\,cos\,\lambda\frac{d\rho}{dt} \qquad (6)$$

As done above we multiply (5) by *cosl* and *cosL* separately to get

$$a\,cos\,L\,cos\,l = b\,cos^2 l - \rho\,cos\,\lambda\,cos\,l \qquad (7)$$

$$a\,cos^2 L = b\,cos\,l\,cos\,L + \rho\,cos\,L\,cos\,\lambda \qquad (8)$$

Adding equations (2) and (6) we get,

$$\rho^2\left(cos^2\lambda + sin^2\lambda\right)\frac{d\lambda}{dt} = b\,\rho\,(cos\,\lambda\,cos\,l + sin\,\lambda\,sin\,l)\,\frac{dl}{dt} - a\,\rho\,(cos\,\lambda\,cos\,L + sin\,\lambda\,sin\,L)\,\frac{dL}{dt}$$

$$\rho^2\frac{d\lambda}{dt} = b\,\rho\,(cos\,\lambda\,cos\,l + sin\,\lambda\,sin\,l)\,\frac{dl}{dt} - a\,\rho\,(cos\,\lambda\,cos\,L + sin\,\lambda\,sin\,L)\,\frac{dL}{dt}$$

$$\rho^2\frac{d\lambda}{dt} = b\,\rho\,cos\,(\lambda - l)\,\frac{dl}{dt} - a\,\rho\,cos\,(\lambda - l)\,\frac{dL}{dt} \qquad (9)$$

adding (3) and (7) we get

$$b = a\,cos\,(l - L) - \rho\,cos\,(\lambda - l) \qquad (10)$$

Similarly adding (4) and (8) we get,

$$a = b\,cos\,(l - L) - \rho\,cos\,(\lambda - l) \qquad (11)$$

Inserting (10) and (11) in to (9) we get,

$$\rho^2\frac{d\lambda}{dt} = ab\,cos\,(l - L)\,\frac{dl}{dt} + b^2\frac{dl}{dt} - a^2\frac{dL}{dt} + ab\,cos\,(l - L)\,\frac{dL}{dt}$$

$$\rho^2\frac{d\lambda}{dt} = \left(a\,b\,cos\,(l - L)\, -\, a^2\right)\frac{dL}{dt} + \left(ab\,cos\,(l - L)\, -\, b^2\right)\frac{dl}{dt} \qquad (12)$$

From Kepler's III law, we know that

$$\left(\frac{dL}{dt}\right)^2 a^3 = \left(\frac{dl}{dt}\right)^2 b^3 = constant = m \qquad (13)$$

which also means

$$\frac{dl}{dt} = \left(\frac{m}{b^3}\right)^{\frac{1}{2}} \qquad (14a)$$

$$\frac{dL}{dt} = \left(\frac{m}{a^3}\right)^{\frac{1}{2}} \qquad (14b)$$

Inserting (14a) and (14b) in (12), we get

$$\rho^2\frac{d\lambda}{dt} = \left(a\,b\,cos\,(l - L)\, -\, a^2\right)\frac{\sqrt{m}}{a\sqrt{a}} + \left(a\,b\,cos\,(l - L)\, -\, b^2\right)\frac{\sqrt{m}}{b\sqrt{b}}$$

Rearranging we get

$$\rho^2 \frac{d\lambda}{dt} = \sqrt{m}\left[\cos(l-L)\left\{\frac{b}{\sqrt{a}} + \frac{a}{\sqrt{b}}\right\} - \left\{\sqrt{a}+\sqrt{b}\right\}\right] \tag{15}$$

This will be zero when

$$\cos(l-L) = \frac{\left\{\sqrt{a}+\sqrt{b}\right\}}{\left\{\frac{b}{\sqrt{a}} + \frac{a}{\sqrt{b}}\right\}} \tag{16}$$

which can be further simplified to

$$\cos(l-L) = \frac{\sqrt{a}\sqrt{b}}{a - \left(\sqrt{a}\sqrt{b}\right) + b} \tag{17}$$

From the figure we see that (l – L) is the angle PSE, which is the difference of longitudes. When this satisfies the condition (17) the planet will be stationary. There are two solutions for a given set of values of a and b. Thus, there are two stationary points and the planet appears to execute retrograde motion in the interval between these points.

2 Alternative procedure

The same result can be obtained by consideration of angular speeds of the earth and the planet which should sum to zero. The same configuration of Figure 1 is used in Figure 2 with the velocities represented by PV and EU. For ease of representation angles SPE and PSE are represented by θ and α whereas the angle PEU is called ψ. We need to get the angular velocity components along the line of sight EP.

The observed angular speed w_{obs} is given by

$$W_{obs} = \frac{v_{rel}}{\rho} \tag{18}$$

where v_{rel} is the velocity of the planet relative to that of earth. The velocity of P along PE is given by VA which is parallel to PE. Similarly, the velocity of E along the same line is given by UB. The perpendicular components are along AP and EB respectively. The relative velocity is $(AP - EB)$.

It can be shown that $VA = v \sin VPA$ and $PA = v \cos VPA$, where v is the velocity.

Therefore

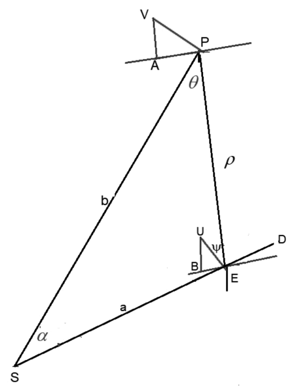

Figure 2 Formulation of the problem by an alternative method.

It can be[1] shown that $VPA = 90–(\alpha + \psi)$
Therefore $VA = v\, cos(\alpha + \psi)$ and

$$PA = v \sin(\alpha + \psi) \tag{19}$$

Similarly considering u as velocity of earch E
$UB = u \sin UE = u \cos\psi$ and

$$BE = u \sin UEA = u \sin\psi \tag{20}$$

$$v_{rel} = v \sin(\alpha+\psi) - u \sin\psi$$

$$W_{obs} = \frac{v_{rel}}{\rho}$$

$$w_{obs} = \frac{v_{obs}}{\rho} = \frac{\{v \sin(\alpha+\psi) - u \sin\psi\}}{\rho}$$

$$= \left\{\frac{v \sin\alpha \cos\psi + v \cos\alpha \sin\psi - u \sin\psi}{\rho}\right\} \tag{21}$$

We have the relations
$a = b\cos\alpha - \rho\cos PES = b\cos\alpha - \rho\sin\psi$, which gives

$$\sin\psi = \frac{(b\cos\alpha - a)}{\rho} \tag{22a}$$

$$\cos\psi = \frac{b\sin\alpha}{\rho} \tag{22b}$$

[1] $VPA = SPE = 180–(PSE + PES) = 180°–(\alpha + SEU + UEP)$
$= 180°–(\alpha + 90 + \psi)$
$= 90–(\alpha + \psi)$

Using (22a) and (22b) we can write (21) as

$$W_{obs} = \frac{v_{rel}}{\rho} = \frac{\{u \sin\alpha \cos\psi + u \cos\alpha \sin\psi - u \sin\psi\}}{\rho}$$

$$w_{obs} = \frac{v \sin\alpha \frac{b \sin\alpha}{\rho} + v \cos\alpha \frac{(b \cos\alpha - a)}{\rho} - u \frac{(b \cos\alpha - a)}{\rho}}{\rho}$$

$$w_{obs} = \frac{\left\{vb \sin^2\alpha - va \cos\alpha + bv \cos^2\alpha + ua - ub \cos\alpha\right\}}{\rho^2}$$

$$w_{obs} = \frac{\{vb + ua - \cos\alpha\,(ub + va)\}}{\rho^2} \qquad (23)$$

When w_{obs} is zero

$$\{vb + ua - \cos\alpha\,(ub + va)\}$$

should be zero. Then,

$$\cos\alpha = \frac{vb + ua}{ub + av} \qquad (24)$$

By using Kepler's laws this can be shown to be equivalent to (17), replacing u and v with powers of a and b, the constant gets eliminated.

Further, (25) can be rewritten using $u = w_e a$ and $v = w_p b$ as,

$$\cos\alpha = \frac{w_e a^2 + w_o b^2}{\{w_o ab + w_e ab\}} \qquad (25)$$

This can be used to get the duration of retrograde motion t, since this angle $\alpha = t\Delta w$, (the difference in angular speeds is Δw) and the orbital parameters are known. Table 1 gives these values.

3 Formula in *Karaṇakutūhala*

In the paper on the analysis of the examples provided in *Brahmatulyaudāharaṇam*, by Visvanatha, (Shubha and Shylaja, 2020) we used the method provided in *Karaṇakutūhala* (Balachandra Rao and Uma, 2008). There we had used ϕ to represent α in Figure 2. The derivation resulted in

$$\cos\phi = \frac{va_j^2 + a^2 u}{aa_j\,(u + v)} \qquad (26)$$

By substituting b for a_j this essentially reduces to conventional formula given in standard texts books of spherical astronomy (see for instance Green 1985).

$$\cos\phi = \frac{\left(a^2 u + b^2 v\right)}{ab\,(u + v)} \qquad (27)$$

Upon application of Kepler's law $u : v = (a/b)^{-3/2}$, this further leads to the conventional formula

$$\cos\phi = \frac{\sqrt{a}\sqrt{b}}{a - \sqrt{a}\sqrt{b} + b} \qquad (28)$$

consider

$$\cos\phi = \frac{\left(a^2 u + b^2 v\right)}{ab\,(u + v)}$$

$$\frac{\left(a^2 u + b^2 v\right)}{ab\,(u + v)}$$

$$= \frac{a^2 \frac{u}{v} + b^2}{ab\frac{u}{v} + ab}$$

$$= \frac{a^2 \left(\frac{a}{b}\right)^{\frac{-3}{2}} + b^2}{ab\left(\frac{a}{b}\right)^{-\frac{3}{2}} + a}$$

$$\frac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{a^{\frac{-1}{2}} b + ab^{\frac{-1}{2}}}$$

$$= \frac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{\frac{b^{\frac{3}{2}} + a^{\frac{3}{2}}}{a^{\frac{1}{2}} b^{\frac{1}{2}}}}$$

$$\frac{a^{\frac{1}{2}} b^{\frac{1}{2}} \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)}{a^{\frac{3}{2}} + b^{\frac{3}{2}}} =$$

$$= \frac{a^{\frac{1}{2}} b^{\frac{1}{2}} \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)}{\left(a^{\frac{3}{2}} + b^{\frac{3}{2}}\right)\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)}$$

$$= \frac{a^{\frac{1}{2}} b^{\frac{1}{2}}\,(a - b)}{a^2 - a^{\frac{3}{2}} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{\frac{3}{2}} - b^2}$$

$$= \frac{a^{\frac{1}{2}} b^{\frac{1}{2}}\,(a - b)}{a^2 - b^2 - a^{\frac{3}{2}} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{\frac{3}{2}}}$$

Table 1 The duration of retrograde motion as derived from equation (25) and compared with values from wikipedia

Planet Name	Orbit **radius (AU)**	**Orbital period (d)**	**From Eq. (25)**	**Actual***
Mercury	0.39	88	22.9	21
Venus	0.72	225	42	41
Mars	1.52	687	72.9	72
Jupiter	5.2	4333	120.7	121
Saturn	9.58	10756	137.5	138
Uranus	19.2	30687	151.7	151
Neptune	30.1	60190	158.4	158
				*Wikipedia

$$= \frac{a^{\frac{1}{2}} b^{\frac{1}{2}} (a-b)}{a^2 - b^2 - a^{\frac{3}{2}} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{\frac{3}{2}}}$$

$$= \frac{a^{\frac{1}{2}} b^{\frac{1}{2}} (a-b)}{((a+b)(a-b)) - \left(a^{\frac{1}{2}} b^{\frac{1}{2}} (a-b)\right)}$$

$$= \frac{a^{\frac{1}{2}} b^{\frac{1}{2}} (a-b)}{(a-b)\left((a+b) - a^{\frac{1}{2}} b^{\frac{1}{2}}\right)}$$

$$= \frac{a^{\frac{1}{2}} b^{\frac{1}{2}}}{\left((a+b) - a^{\frac{1}{2}} b^{\frac{1}{2}}\right)}$$

$$= \frac{\sqrt{a}\sqrt{b}}{a - \sqrt{a}\sqrt{b} + b}$$

Thus, we have

$$\frac{\left(a^2 u + b^2 v\right)}{ab\,(u+v)} = \frac{\sqrt{a}\sqrt{b}}{a - \sqrt{a}\sqrt{b} + b}$$

4 Conclusion

The formulae for the stationary points which cause an apparent retrograde motion as seen from the earth are derived using different methods. The derivations are provided from different methods and compared with the one used in earlier Indian astronomical texts.

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