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Mādhava's Multi-pronged Approach for Obtaining the *Prāṇakalāntara*

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Abstract

The *prāṇakalāntara*, or the difference between the longitude and the corresponding right ascension, is an important astronomical parameter used in determining the ascendant (*lagna*), as well as the equation of time in Indian astronomy. This paper explores the different algorithms described to calculate the *prāṇakalāntara* in the *Lagnaprakaraṇa*, a hitherto unpublished manuscript attributed to Mādhava, the founder of the Kerala school of astronomy and mathematics. We also point out the interpretation of some of the algorithms in terms of epicyclic models.

Key words: *Asu*, *Dyujyā*, Epicyclic models, *Kalāsubheda*, *Lagnaprakaraṇa*, *Liptā*, *Liptāsubheda*, Mādhava, *Prāṇakalāntara*

1. INTRODUCTION

India has a long and rich history of the study of astronomy, which extends back a few millennia. Renowned astronomer-mathematicians like Āryabhaṭa, Bhāskara I, Brahmagupta, Lalla, Śrīdhara, Śrīpati, Bhāskara II etc. have made numerous contributions to the study of astronomy as well as mathematics. Starting in the fourteenth century CE, a succession of scholars enjoying a teacher-taught lineage (*guru-śiṣya-parampara*), and collectively referred to as the Kerala school of astronomy and mathematics,[1] made enormous contributions to the study of astronomy as well as mathematics. Mādhava of Saṅgamagrāma, the illustrious founder of this school is credited with a number of important results like an infinite series for π, as well as the functions sine and cosine. He is credited with a number of astronomical works including *Veṇvāroha*, *Sphuṭacandrāpti*, *Agaṇita-grahacāra*, *Candravākyāni*, *Madhyamānayanaprakāra*, *Mahājyānayanaprakāra*, *Lagnaprakaraṇa*, and perhaps *Golavāda*.[2]

The *Lagnaprakaraṇa*[3] is a work dedicated to the determination of the ascendant or the *udayalagna*, and discuses numerous techniques for the same. However, as a necessary precursor to determining the ascendant, the text first discusses various methods to determine the *prāṇakalāntara*, which is the difference between the longitude and corresponding right ascension of a star.

In this paper, we first discuss the *prāṇakalāntara* and its significance in Section 2. In Section 3, we discuss the different algorithms for de-

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[1] So named because all these scholars resided in the Malabar coast of the Kerala state in India.

[2] See Sarma 1977. Also see Pingree 1981, pp. 414–415.

[3] The authors obtained two manuscripts of the *Lagnaprakaraṇa* from the Prof. K. V. Sarma Research Foundation, Chennai.

termining the *prāṇakalāntara* as described in the *Lagnaprakaraṇa*, and also interpret a few of the algorithms in terms of epicyclic models. Finally, in Section 4, we discuss the significance of these different techniques, and conclude with our remarks.

2. *PRĀṆAKALĀNTARA* AND ITS SIGNIFICANCE

The word *prāṇakalāntara* is a compound word that is made up on three parts: *prāṇa*, *kalā*, and *antara*.[4] The term *prāṇa* generally refers to a certain unit of time that is close to four seconds, and is equal to the unit of time corresponding to one arc-minute of the celestial equator.[5] This unit of time is also sometimes referred to as an *asu*. The term *kalā* refers to one-sixtieth part of a degree. In the current context, it refers to a unit of measure along the ecliptic equal to one arc-minute. The word *antara* in Sanskrit means 'difference'. Hence, the compound word *prāṇakalāntara* is used as a technical term denoting the difference between the longitude and corresponding right ascension.[6] In *Lagnaprakaraṇa*, *prāṇakalāntara* is primarily used to convert the longitude of any point on the ecliptic into its corresponding right ascension, or vice-versa. Nīlakaṇṭha also employs the *prāṇakalāntara* to determine the equation of time in his *Tantrasaṅgraha*.[7]

Denoting the longitude with λ, and the right ascension with α, the *prāṇakalāntara* can be expressed in mathematical notation as

$$prāṇakalāntara = \lambda - \alpha.$$

While this quantity can be positive or negative depending on the point of the ecliptic under consideration, Indian astronomers considered only its magnitude, and applied it positively or negatively to the longitude to obtain the right ascension. Therefore, the general formula for determining the right ascension, given the longitude and the *prāṇakalāntara*, can be written as follows:

$$\alpha = \lambda \pm prāṇakalāntara = \lambda \pm |\lambda - \alpha|\,.$$

3. DETERMINATION OF *PRAṆAKALĀNTARA* IN THE *LAGNAPRAKARAṆA*

The *Lagnaprakaraṇa* can be divided into eight chapters, of which the last seven chapters are essentially dedicated to different methods of determining the *lagna*. The first chapter lays the foundation for this by introducing various concepts like declination, *prāṇakalāntara*, *dyujyā* or radius of diurnal circle, ascensional difference, and *kālalagna* or the time elapsed since the rise of the vernal equinox at a desired instance. All these variables have great physical significance. However, depending on the approach to the problem, only a select few of them would appear in a specific procedure of calculating *lagna*.

After the customary invocation, and after quickly defining the declination and versine, the text immediately jumps into the discussion of *prāṇakalāntara* in verses 6–17, which describe various methods of calculating this quantity. Below, we discuss the different methods to determine the *prāṇakalāntara* discussed in the *Lagnaprakaraṇa*. In the following discussion, it may be noted that the radius of the diurnal circle of a given body having declination of δ is nothing but $R\cos\delta$, where R is the radius of the equator. It may also be noted that verses 13 and 14 discuss the appropriate sign to be used with the *prāṇakalāntara*, depending upon the quadrant of the ecliptic under consideration.

[4] The first two words are combined through a *karmadhāraya* compound, and the last one through a *ṣaṣṭhītatpuruṣa*.

[5] The equator consists of $360*60 = 21600$ arc-minutes, while the period of Earth's rotation is approximately $24*60*60 = 84600$ seconds. Therefore, one arc-minute of the equator would take approximately 4 seconds to cut across the prime meridian.

[6] It may be noted that both *prāṇas* and *kalās* are measured in the same unit essentially, as both correspond to one-sixtieth of a degree, though measured along the equator and the ecliptic respectively.

[7] See Ramasubramanian and Sriram 2011, pp. 80–82.

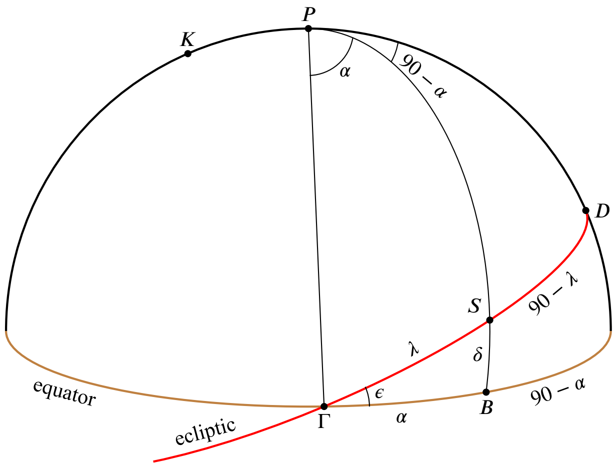


Fig. 1. Determining the *prāṇakalāntara*.

Method 1

अन्त्यद्युजीवाहतबाहुजीवाम्
इष्टद्युमौर्व्या विभजेदवाप्तम् ।
चापीकृतं बाहुगुणस्य चापात्
विशोधितं प्राणकलान्तरं स्यात् ॥६॥

antyadyujīvāhatabāhujīvām
iṣṭadyumaurvyā vibhajedavāptam |
cāpīkṛtaṃ bāhuguṇasya cāpāt
viśodhitaṃ prāṇakalāntaraṃ syāt ||6||

One should divide the Rsine [of the Sun's longitude] (*bāhujīvā*)—which is multiplied by the last radius of the diurnal circle (*antyadyujīvā*)—by the given radius of the diurnal circle (*dyumaurvī*). The quotient converted to arc [minutes] and then subtracted from the arc corresponding to the Rsine [of the Sun's longitude] would be the difference of the longitude and right ascension (*prāṇakalāntara*).

The expression for *prāṇakalāntara* given in the above verse can be understood with the help of Fig. 1. Here, P and K represent the poles of the equator and ecliptic respectively. Γ represents the vernal equinox, while S is the Sun with longitude λ and declination δ. ϵ denotes the obliquity of ecliptic, which is also the maximum possible declination of the Sun. The meridian passing through the Sun intersects the equator at B, and therefore ΓB represents the right ascension α of the Sun. Using the above notations, the expression for *prāṇakalāntara* given in the verse may be expressed as:

$$\lambda - \alpha = \lambda - R\sin^{-1}\left(\frac{R\sin\lambda \times R\cos\epsilon}{R\cos\delta}\right). \quad (1)$$

The second expression on the right-hand side can be easily seen to be the right ascension of the Sun by considering the spherical triangle $P\Gamma S$. Here, $PS = 90 - \delta$ and $P\hat{\Gamma}S = 90 - \epsilon$. Now, applying the sine rule of spherical trigonometry, we have

$$\frac{\sin\alpha}{\sin\lambda} = \frac{\sin(90-\epsilon)}{\sin(90-\delta)},$$

$$\text{or,} \qquad \alpha = \sin^{-1}\left(\frac{\sin\lambda\cos\epsilon}{\cos\delta}\right). \quad (2)$$

In the above expression, the sines are dimensionless, whereas the Indian Rsine is a linear measure. Taking this into account, the expression for α may be written as:

$$\alpha = R\sin^{-1}\left(\frac{R\sin\lambda \times R\cos\epsilon}{R\cos\delta}\right).$$

Thus, we easily see that (1) is nothing but the difference of the longitude and the right ascension, i.e. *prāṇakalāntara*.

Method 2

कोटीगुणं व्यासदलेन संह-
त्येष्टद्युमौर्व्या प्रविभज्य लब्धात् ।
चापीकृतात् कोटिगुणस्य चापे
त्यक्तेऽथवा प्राणकलान्तरं स्यात् ॥७॥

koṭīguṇaṃ vyāsadalena saṃha-
tyeṣṭadyumaurvyā pravibhajya labdhāt |
cāpīkṛtāt koṭiguṇasya cāpe
tyakte'thavā prāṇakalāntaraṃ syāt ||7||

Or, the *prāṇakalāntara* would be that when the arc of the Rcosine [of the Sun's longitude] (*koṭiguṇa*) is subtracted from the arc of the quotient obtained after multiplying the Rcosine [of the Sun's longitude] (*koṭīguṇa*) by the semi-diameter, [and then] dividing it by the day-radius (*dyumaurvī*).

The expression coded in the above verse to determine the *prāṇakalāntara* can be expressed as follows:

$$\lambda-\alpha = R\sin^{-1}\left(\frac{R\cos\lambda\times R}{R\cos\delta}\right)-R\sin^{-1}(R\cos\lambda). \tag{3}$$

The rationale behind the above expression can be understood as follows. As indicated in Fig. 1, the Sun reaches its maximum declination at point D on the prime meridian. Now, considering the spherical triangle PSD, and applying the sine rule, we have

$$\frac{\sin(90-\alpha)}{\sin(90-\lambda)} = \frac{\sin 90^\circ}{\sin(90-\delta)},$$

$$\text{or,}\qquad \sin(90-\alpha) = \frac{\cos\lambda}{\cos\delta}. \tag{4}$$

It can be seen that the sine inverse of the right-hand side of the above expression is equivalent to the first term in the RHS of (3). Now, subtracting the sine inverse of the cosine of the longitude of the Sun from this quantity, we obtain the *prāṇakalāntara*:

$$\begin{aligned}&\sin^{-1}\left(\frac{\cos\lambda}{\cos\delta}\right)-\sin^{-1}\cos\lambda\\ &=\sin^{-1}\sin(90-\alpha)-\sin^{-1}\sin(90-\lambda)\\ &=\lambda-\alpha.\end{aligned}$$

Method 3

अन्त्यक्रान्तिशराहताद्भुजगुणात् त्रिज्याप्तमन्त्यं फलं
विद्याद्बाहुगुणात्तदन्तिमफलं संशोध्य वर्गीकृतात् ।
कोटिज्याकृतिसंयुतात् पदमिह द्युज्या तया संहरेत्
कोटिज्यान्तफलाहतिं फलमिदं लिप्तासुभेदं विदुः ॥८॥

antyakrāntiśarāhatādbhujaguṇāt trijyāpta-mantyaṃ phalaṃ
vidyādbāhuguṇāttadantimaphalaṃ saṃśodhya vargīkṛtāt |
koṭijyākṛtisaṃyutāt padamiha dyujyā tayā saṃharet
koṭijyāntaphalāhatiṃ phalamidaṃ liptāsubhedaṃ viduḥ ||8||

One should know the *antyaphala* as the quotient obtained from dividing the product of the Rsine [of the Sun's longitude] (*bhujaguṇa*) and the versine corresponding to the maximum declination (*antyakrāntiśara*) by the radius (*trijyā*). Having subtracted that *antimaphala* (*antyaphala*) from the Rsine [of the Sun's longitude], the square-root taken from square [of that quantity] added by the square of the Rcosine [of the Sun's longitude] (*koṭijyā*), is the radius of the diurnal circle (*dyujyā*). One should divide the product of the *koṭijyā* and the *antyaphala* by that (*dyujyā*). [Scholars] know this quotient as the difference of the [Sun's] longitude and right ascension (*liptāsubheda*).

This verse first defines a quantity known as *antyaphala* as follows:

$$\textit{antyaphala} = \frac{R\sin\lambda\times R\,\mathrm{versin}\,\epsilon}{R}. \tag{5}$$

It then presents relations for the radius of the diurnal circle (*dyujyā* or $R\cos\delta$), and the *prāṇakalāntara*. The expressions for these two quantities presented in the text may be expressed as

$$R\cos\delta = \sqrt{(R\sin\lambda-\textit{antyaphala})^2+(R\cos\lambda)^2}, \tag{6}$$

$$\lambda-\alpha = \frac{R\cos\lambda\times\textit{antyaphala}}{R\cos\delta}. \tag{7}$$

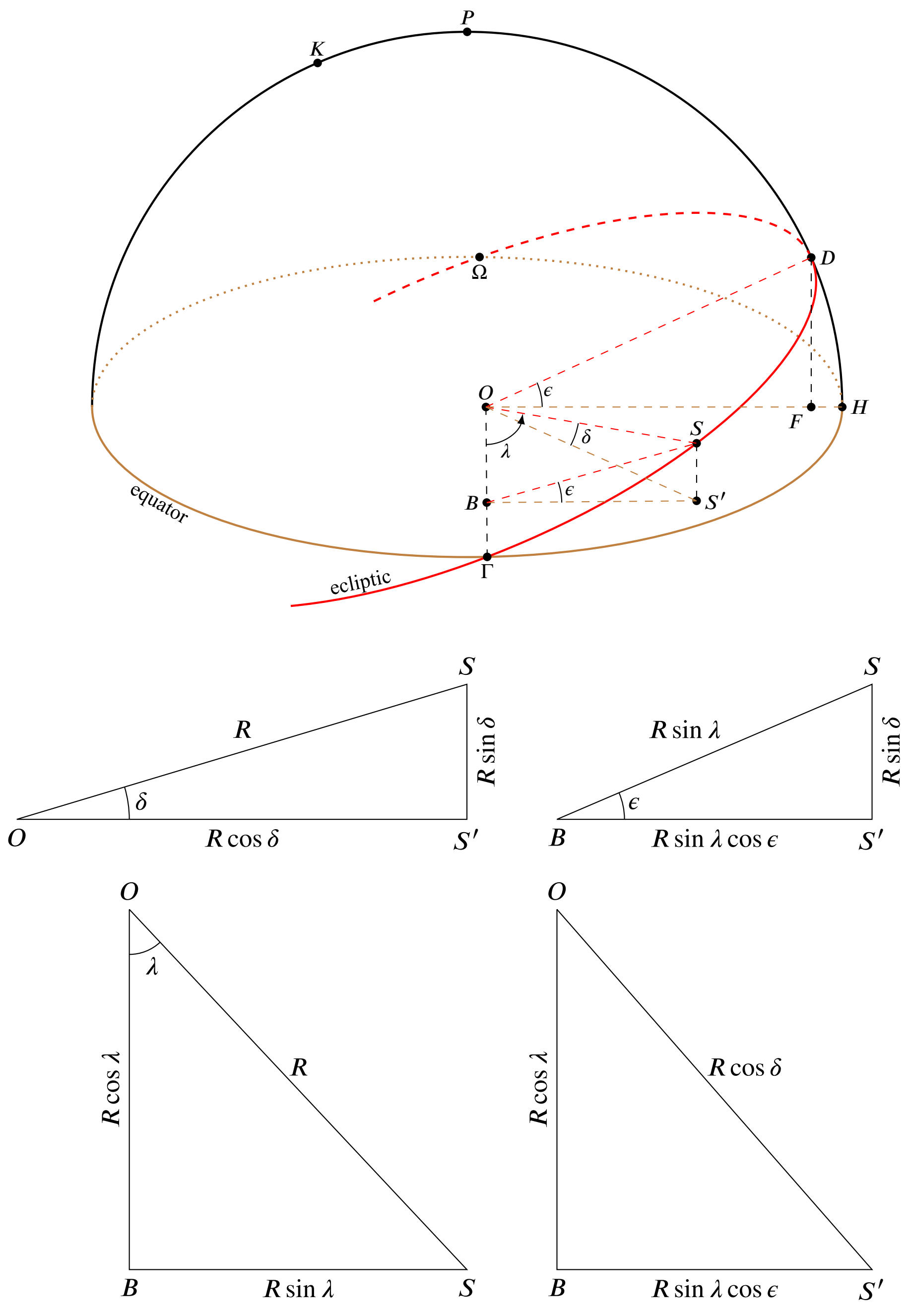


Fig. 2. Determining the *antyaphala* and *dyujyā*.

The rationale behind (6) can be understood from Fig. 2. In the figure, S and S' indicate a point on the ecliptic, and its projection on the equatorial plane respectively. The planar right-angled triangle SOS' lies in the meridian plane passing through S, along which the declination is measured. Therefore, we have $S\hat{O}S' = \delta$, and hence

$$SS' = R\sin\delta, \qquad OS' = R\cos\delta.$$

The right-angled[8] triangle OBS lies in the plane of the ecliptic. The angle $B\hat{O}S = \lambda$, and hence

$$BS = R\sin\lambda, \qquad OB = R\cos\lambda.$$

The right-angled triangle SBS' lies in a plane perpendicular to the equator, and is parallel to the plane containing the great circle passing through the poles of the ecliptic (K) and equator (P). Hence, the angle $S\hat{B}S' = \epsilon$, and

$$BS = R\sin\lambda, \qquad BS' = R\sin\lambda\cos\epsilon.$$

Now, consider the right-angled triangle OBS' that lies on the plane of the equator, and whose sides have been calculated above. Since $OS'^2 = BS'^2 + OB^2$, we have

$$\begin{aligned} R\cos\delta &= \sqrt{(R\sin\lambda\cos\epsilon)^2 + (R\cos\lambda)^2} \\ &= \sqrt{(R\sin\lambda - \textit{antyaphala})^2 + (R\cos\lambda)^2}, \end{aligned}$$

which is the same as (6) given in the text.[9]

Thus, we find that the expression for *dyujyā* given by (6) is an exact expression. However, the expression for the *prāṇakalāntara* which is given in terms of the *antyaphala* and the *dyujyā* in (7) seems to be approximate, and can be understood as follows. Substituting (5) in (7), and employing $\text{versin}\,\epsilon = 1 - \cos\epsilon$, we have

$$\frac{\cos\lambda \times \textit{antyaphala}}{\cos\delta} = \frac{\cos\lambda\sin\lambda}{\cos\delta} - \frac{\cos\lambda\sin\lambda\cos\epsilon}{\cos\delta}.$$

Substituting

$$\cos\alpha = \frac{\cos\lambda}{\cos\delta}, \qquad \sin\alpha = \frac{\sin\lambda\cos\epsilon}{\cos\delta},$$

from (4) and (2) respectively, the above expression reduces to[10]

$$\begin{aligned} \frac{\cos\lambda \times \textit{antyaphala}}{\cos\delta} &= \sin\lambda\cos\alpha - \cos\lambda\sin\alpha \\ &= \sin(\lambda - \alpha). \end{aligned}$$

This appears to have been approximated as $\lambda - \alpha$. Indian mathematicians knew at least from the time of Āryabhaṭa that $\sin\theta \approx \theta$, for small θ. The *prāṇakalāntara* has a maximum value of approximately 2.6°, at a longitude of 46°, as shown in Fig. 3.[11] Therefore, the use of this approximation is justified.

From a geometric point of view, the expression for the *antyaphala* can be thought of as the difference between the hypotenuse and the base of the triangle SBS'. That is

$$\begin{aligned} BS - BS' &= R\sin\lambda - R\sin\lambda\cos\epsilon \\ &= R\sin\lambda\,\text{versin}\,\epsilon, \end{aligned}$$

[8]The triangle does not appear right-angled in the figure due to the difficulty in depicting the three-dimensional celestial sphere on a two-dimensional surface.

[9]It may be mentioned here that while discussing the determination of right ascension, Nīlakaṇṭha in his *Tantrasaṅgraha* employs the term *koṭikā* to refer to the simplified expression

$$R\sin\lambda\cos\epsilon = R\sin\lambda - \textit{antyaphala}.$$

For details, see Ramasubramanian and Sriram 2011, pp. 76–79.

[10]In the *Tantrasaṅgraha*, Nīlakaṇṭha indicates that Mādhava was aware of the rule

$$\sin(A - B) = \sin A\cos B - \cos A\sin B.$$

See Ramasubramanian and Sriram 2011, pp. 70–73.

[11]Plotted taking the obliquity of the ecliptic as $\epsilon = 24°$.

Longitude (λ°)	*prāṇakalāntara* ($[\lambda - \alpha]^\circ$)
0	0
15	1.245
30	2.191
45	2.587
60	2.292
75	1.347
90	0

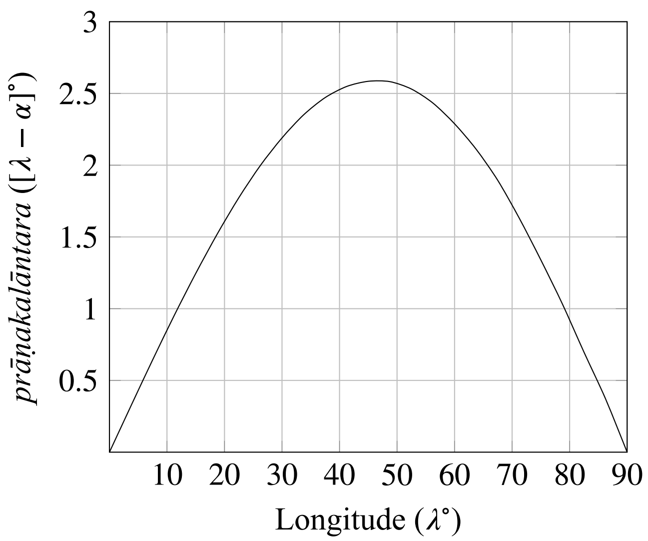


Fig. 3. Variation of *prāṇakalāntara* with longitude.

which is essentially the same as (5). When the Sun's longitude is 90 at D, this expression reaches its maximum value, which can also be deduced from the triangle DOF:

$$\begin{aligned} R \sin\lambda \operatorname{versin}\epsilon &= R \operatorname{versin}\epsilon \\ &= R - R\cos\epsilon \\ &= OD - OF. \end{aligned}$$

Method 4

दोःकोटिजीवे पुनरन्तिमेन
फलेन हत्वा विभजेत् त्रिमौर्व्या ।
लब्धं भुजाकोटिफले पुनस्तत्
भुजाफलं व्यासदलाद्विशोध्यम् ॥९॥
शिष्टस्य कोट्याश्च फलस्य कृत्योः
समासमूलं भवति द्युजीवा ।
कोटीफलं व्यासदलेन हत्वा
द्युज्याहृतं प्राणकलान्तरं स्यात् ॥१०॥

doḥkoṭijīve punarantimena
phalena hatvā vibhajet trimaurvyā |
labdhaṃ bhujākoṭiphale punastat
bhujāphalaṃ vyāsadalādviśodhyam ||9||
śiṣṭasya koṭyāśca phalasya kṛtyoḥ
samāsamūlaṃ bhavati dyujīvā |
koṭīphalaṃ vyāsadalena hatvā
dyujyāhṛtaṃ prāṇakalāntaraṃ syāt ||10||

Again, having multiplied the Rsine and Rcosine [of the Sun's longitude] by *antimaphala* (*antyaphala*), one should divide [the obtained

> quantities] by the radius (*trimaurvī*). The quotients obtained are *bhujāphala* and *koṭiphala*. Again, that *bhujāphala* should be subtracted from the semi-diameter. The square-root of the sum of the squares of the residue and *koṭiphala* is the radius of the diurnal circle (*dyujīvā*). The *koṭīphala* multiplied by the semi-diameter and divided by *dyujyā* would be *prāṇakalāntara*.

These two verses describe yet another way of calculating (i) the radius of the diurnal circle, and (ii) the *prāṇakalāntara*, with the help of two intermediary quantities

$$bhuj\bar{a}phala = \frac{R\sin\lambda \times antyaphala}{R}, \quad (8)$$

$$ko\d{t}\bar{\imath}phala = \frac{R\cos\lambda \times antyaphala}{R}. \quad (9)$$

Then, the radius of the diurnal circle is given as

$$R\cos\delta = \sqrt{(R - bhuj\bar{a}phala)^2 + (ko\d{t}\bar{\imath}phala)^2}, \quad (10)$$

and the *prāṇakalāntara* as

$$\lambda - \alpha = \frac{ko\d{t}\bar{\imath}phala \times R}{R\cos\delta}. \quad (11)$$

Expressions (10) and (11) can be shown to be equal to results obtained earlier through simple trigonometric manipulation. Readers can easily verify that (10) indeed yields (6) upon substituting (8) and (9) in it, and that (11) reduces to (7) upon substituting (9) in it.[12]

However, the above relations also have a deep geometric significance. The author appears to have conceived of a geometric model akin to the classical epicycle or *nīcoccavṛtta* model employed in determining the true position of a planet from its mean position. In this model, the radius of the epicycle—which is taken to be *antyaphala* here—is a function of the sine of the longitude of the Sun and hence would be zero at $\lambda = 0$ or 180, and will be maximum at $\lambda = 90$ or 270. Fig. 4 depicts this model wherein, the radius of the epicycle is given by

$$a = P_0P = R\sin\lambda\,\mathrm{versin}\,\epsilon = antyaphala,$$

while $OP_0 = R$ is the radius of the deferent circle. The radius of the diurnal circle $OP = R\cos\delta$ is what is to be determined. It is easily seen that the radius of the diurnal circle is maximum ($= R$) when $\lambda = 0$,[13] as in this case the *antyaphala* would be zero, and both P and P_0 would coincide with X.[14] As the longitude of the Sun increases, the *dyujyā* starts decreasing, and would be shortest when $\lambda = 90$. In this case, the radius of the epicycle (that corresponds to the *antyaphala*) reaches its maximum value which is

$$a_{max} = R - R\cos\epsilon.$$

When this happens, P_0 would coincide with Y and P with T.

In the triangle P_0OP, PQ is a perpendicular dropped on OP_0, and since PP_0 is parallel to OY, we have

$$P\hat{P}_0Q = \lambda' = 90 - \lambda.$$

Then, in the right-angled triangle P_0QP we have

$$\begin{aligned} bhuj\bar{a}phala \quad P_0Q &= \sin\lambda \times antyaphala, \\ ko\d{t}\bar{\imath}phala \quad PQ &= \cos\lambda \times antyaphala, \end{aligned}$$

which are essentially the expressions (8) and (9) given in the text respectively. This also yields

$$OQ = R - bhuj\bar{a}phala.$$

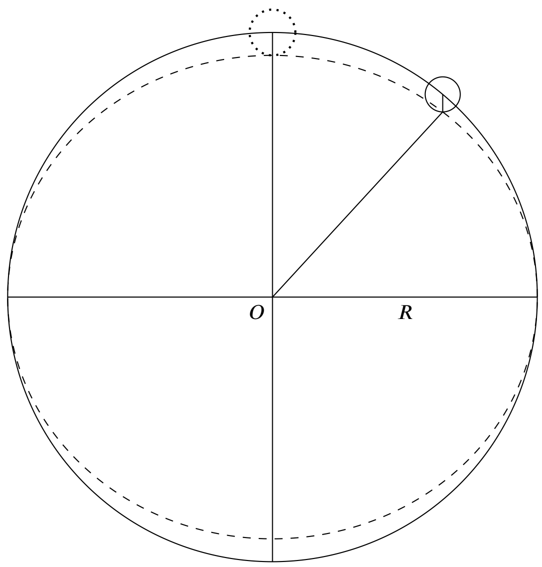
Therefore, in right-angled triangle POQ we have

$$R\cos\delta = \sqrt{(R - bhuj\bar{a}phala)^2 + (ko\d{t}\bar{\imath}phala)^2},$$

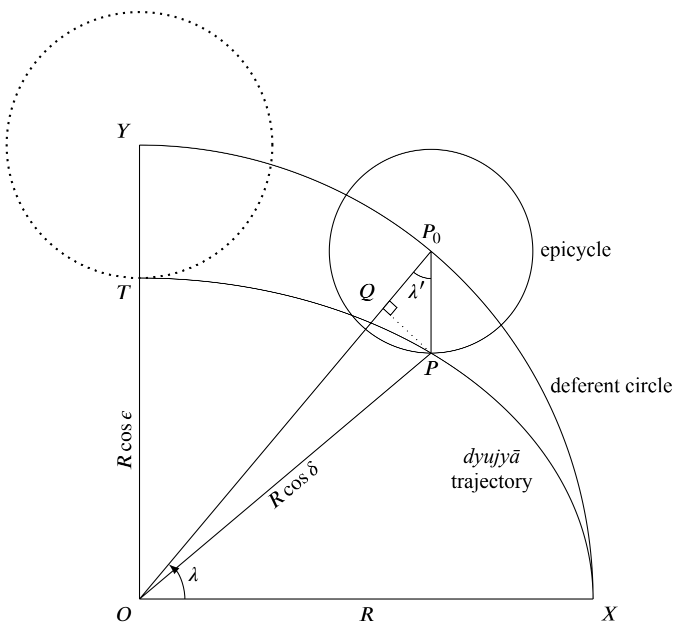
[12] Here again, the approximation $\sin(\lambda - \alpha) \approx \lambda - \alpha$ is used.

[13] This is expected as the declination δ would also be zero, implying a position on the equator.

[14] The deferent circle here is analogous to the equatorial plane, as R is the maximum possible radius ($R\cos\delta$) of the diurnal circle.



(a) The annual trajectory (to scale) of *dyujyā* when mapped on to the equatorial plane.



(b) A quadrant of the trajectory (not to scale).

Fig. 4. Determination of *dyujyā* conceiving an epicyclic model.

which is the same as (10). Therefore, this computation is akin to the computation of the *mandaphala* or *śīghraphala* in the Indian epicycle or eccentric models, with the *dyujyā* playing the role of the *karṇa*. In this specific case, the *karṇa* is always less than or equal to R.

Method 5

यद्वा परक्रान्तिशरेण हत्वा
कोटीगुणं व्यासदलेन हृत्वा ।
लब्धेन दोःकोटिगुणौ निहत्य
त्रिज्याहृते तत्र फले भवेताम् ॥११॥
कोटीफलं क्षिपतु तत्परमद्युमौर्व्यां
तद्वर्गबाहुफलवर्गसमाससमूलम् ।
द्युज्या भवेद्भुजफलाहतविस्तरार्धं[15]
द्युज्याहृतं भवति तत्र कलासुभेदम् ॥१२॥

yadvā parakrāntiśareṇa hatvā
koṭīguṇaṃ vyāsadalena hṛtvā |
labdhena doḥkoṭiguṇau nihatya
trijyāhṛte tatra phale bhavetām ||11||
koṭīphalaṃ kṣipatu tatparamadyumaurvyāṃ
tadvargabāhuphalavargasamāsamūlam |
dyujyā bhavedbhujaphalāhatavistarārdhaṃ
dyujyāhṛtaṃ bhavati tatra kalāsubhedam ||12||

Or, when the Rcosine [of the Sun's longitude] multiplied by the Rversine of the maximum declination and divided by the semi-diameter, [and the result separately] multiplied by the Rsine and Rcosine [of the Sun's longitude] and divided by the radius (*trijyā*), there would be two *phalas* (*bhujaphala* and *koṭīphala*). Add the *koṭīphala* to that last radius of the diurnal circle (*dyumaurvī*). The square-root of the sum of the square of that [previously determined sum] and the square of the *bāhuphala* (*bhujaphala*) would be the radius of the diurnal circle (*dyujyā*). The *bhujaphala* multiplied by the semi-diameter and divided by the *dyujyā* would be the difference in longitude and right ascension (*kalāsubheda*) there.

This verse yet again defines two intermediary quantities *bhujaphala* and *koṭīphala* (different from those in the previous verse):

$$bhujaphala = \frac{R\cos\lambda \times R\,\mathrm{versin}\,\epsilon}{R} \times \frac{R\sin\lambda}{R}, \tag{12}$$

$$ko\d{t}\bar{\imath}phala = \frac{R\cos\lambda \times R\,\mathrm{versin}\,\epsilon}{R} \times \frac{R\cos\lambda}{R}, \tag{13}$$

in service of determining (i) the radius of the Sun's diurnal circle

$$R\cos\delta = \left[(R\cos\epsilon + ko\d{t}\bar{\imath}phala)^2 + (bhujaphala)^2\right]^{\frac{1}{2}}, \tag{14}$$

and (ii) the *prāṇakalāntara*, i.e.

$$\lambda - \alpha = \frac{bhujaphala \times R}{R\cos\delta}. \tag{15}$$

It is easy to verify that, after some basic trigonometric manipulation, (14) yields (6) upon substituting (12)–(13) in it, while (15) yields (7) upon substituting (12) in it.[16]

Here, the author appears to have come up with the given relations by conceiving a different epicyclic model to the one discussed in the previous method. Here, firstly, one has to consider a deferent circle of radius $R\cos\epsilon$, which is the smallest possible radius of the diurnal circle. Secondly, the radius of the epicycle is taken as $R\cos\lambda\,\mathrm{versin}\,\epsilon$ instead of $R\sin\lambda\,\mathrm{versin}\,\epsilon$ or *antyaphala* as in the previous method.

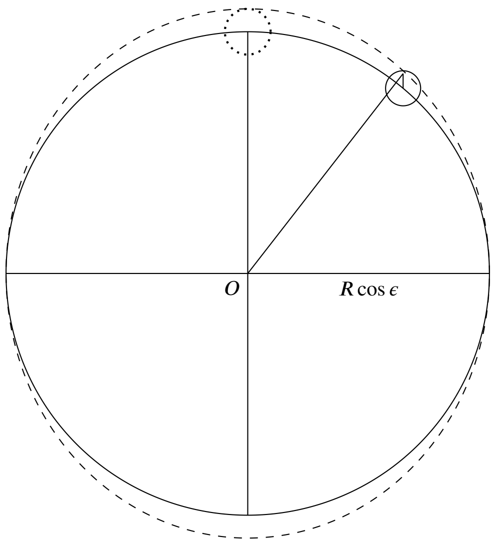
Fig. 5 depicts this model wherein, the radius of the epicycle is given by

$$a' = P_0P = R\cos\lambda\,\mathrm{versin}\,\epsilon,$$

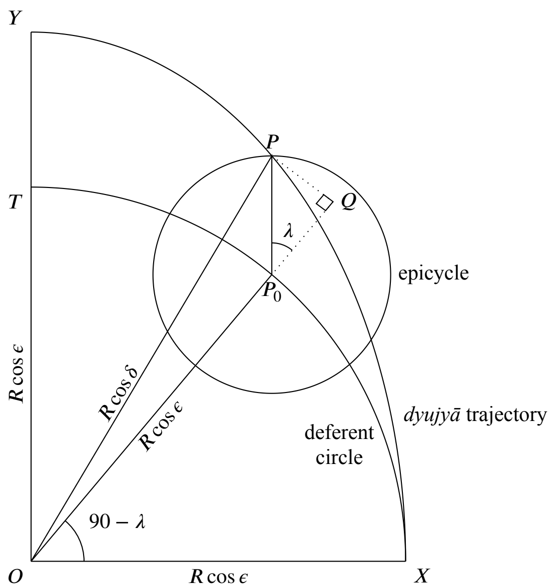
while $OP_0 = R\cos\epsilon$ is the radius of the deferent circle. The radius of the diurnal circle $OP = R\cos\delta$ is what is to be determined. It is easily seen that the radius of the diurnal circle is maximum ($= R$) when $\lambda = 0$, as in this case the radius of the epicycle $a'_{max} = R - R\cos\epsilon$, and P and P_0

[15] Manuscripts read विस्तरार्धा. However, विस्तरार्धं is more appropriate here.

[16] Here again, the approximation $\sin(\lambda - \alpha) \approx \lambda - \alpha$ is used.



(a) The annual trajectory (to scale) of *dyujyā* when mapped on to the equatorial plane.



(b) A quadrant of the trajectory (not to scale).

Fig. 5. Determination of *dyujyā* conceiving an epicyclic model.

would coincide with Y and T respectively. As the longitude of the Sun increases, the *dyujyā* starts decreasing, and would be shortest when $\lambda = 90$. In this case, the radius of the epicycle becomes zero, as both P_0 and P coincide with X.

In the triangle P_0OP, PQ is a perpendicular dropped on extended OP_0, and since P_0P is parallel to OY, we have $P\hat{P}_0Q = \lambda$. Then, in the right-angled triangle P_0QP we have

$$\begin{aligned} \textit{bhujaphala} \quad PQ &= \sin\lambda \times a', \\ \textit{koṭīphala} \quad P_0Q &= \cos\lambda \times a', \end{aligned}$$

which are essentially the expressions (12) and (13) given in the text respectively. This also yields

$$OQ = R\cos\epsilon + \textit{koṭīphala}.$$

Therefore, in right-angled triangle POQ we have

$$R\cos\delta = \left[(R\cos\epsilon + \textit{koṭīphala})^2 + (\textit{bhujaphala})^2\right]^{\frac{1}{2}},$$

which is the same as (14).

Application of *prāṇakalāntara*

The following two verses describe when the *prāṇakalāntara* is to be applied positively or negatively.

दोःकोटिमौर्व्योर्वधतस्त्रिमौर्व्या
लब्धं परापक्रमबाणनिघ्नम् ।
द्युज्याहृतं प्राणकलान्तरं तत्
युग्मौजपादक्रमतो धनर्णम् ॥१३॥

doḥkoṭimaurvyorvadhatastrimaurvyā
labdhaṃ parāpakramabāṇanighnam |
dyujyāhṛtaṃ prāṇakalāntaraṃ tat
yugmaujapādakramato dhanarṇam ||13||

The quotient obtained from the division of the product of the Rsine and Rcosine [of the Sun's longitude] by radius, multiplied by Rversine corresponding to the maximum declination and divided by the radius of the diurnal circle is *prāṇakalāntara*. That is positive and negative depending on even and odd quadrants respectively.

This verse gives the following relation for the *prāṇakalāntara*:

$$\lambda - \alpha = \frac{R\sin\lambda \times R\cos\lambda}{R} \times \frac{R\,\mathrm{versin}\,\epsilon}{R\cos\delta}. \qquad (16)$$

This is just a restatement of the (7), with only the order of terms changed.

The verse also states that the *prāṇakalāntara* is to be applied negatively when the Sun is in the first and third quadrants,[17] and positively when it is in the second and fourth quadrants. This can be understood from the fact that, for the Sun, $\lambda > \alpha$ in the first and third quadrants, and $\lambda < \alpha$ in the second and fourth quadrants.[18] Therefore, the *prāṇakalāntara* of the form $|\lambda - \alpha|$ has to be subtracted from the longitude of the Sun in the first and third quadrants, and added to the longitude of the Sun in the second and fourth quadrants to obtain the correct right ascension.[19]

द्विघ्नस्य सायनरवेर्भुजमौर्विकार्धात्
अन्त्यापमेषुगुणिताद्द्युगुणेन लब्धम् ।
लिप्तासुभेद इह स द्विगुणस्य भानोः
जूकक्रियादिवशतः क्रमशो धनर्णम् ॥१४॥

dvighnasya sāyanaraverbhujamaurvikārdhāt
antyāpameṣuguṇitāddyuguṇena labdham |
liptāsubheda iha sa dviguṇasya bhānoḥ
jūkakriyādivaśataḥ kramaśo dhanarṇam ||14||

[17] Measuring eastwards from the vernal equinox.

[18] For instance, consider triangle $P\Gamma B$ in Fig. 1, where the Sun is depicted in the first quadrant. Here, ΓB will be the shortest great circle arc from Γ to any point on PB as P itself is its pole. Therefore, the great circle arc ΓS, whose pole lies at K, will be longer than ΓB. Hence, we can show that $\lambda > \alpha$ when the Sun is in the first quadrant. Similarly, we can also show that $\lambda > \alpha$ in the third quadrant, and $\lambda < \alpha$ in the second and fourth quadrants.

[19] Indian mathematicians and astronomers typically preferred not to deal with negative numbers. They therefore considered only the absolute value of any difference, and changed the sign of the quantity appropriately during its application in a mathematical operation.

> The quotient obtained from the division of—half of the Rsine of twice the precession corrected longitude of the Sun multiplied by the Rversine corresponding to the last declination—by the radius of the diurnal circle (*dyuguṇa*) is the difference in the longitude and right ascension (*liptāsubheda*) here. That is positive or negative depending on if twice the longitude is [in the six signs] commencing from Libra (*Jūka*) or Aries (*Kriya*) respectively.

This verse gives the following relation for the *prāṇakalāntara*, i.e.

$$\lambda - \alpha = \frac{\frac{1}{2} \times R \sin 2\lambda \times R \operatorname{versin} \epsilon}{R \cos \delta}, \qquad (17)$$

which reduces to (7) upon substituting

$$\sin 2\lambda = 2 \sin \lambda \cos \lambda$$

in it.

The verse states that the *prāṇakalāntara* is to be applied positively when twice the Sun's longitude is in the range 180° to 360° (i.e. $90^\circ < \lambda < 180^\circ$), and negatively when the same quantity is in the range of 0° to 180° (i.e. $0^\circ < \lambda < 90^\circ$).[20] This is equivalent to the statement in the previous verse that *prāṇakalāntara* is to be applied positively when the Sun is in the second and fourth quadrants, and negatively when it is present in the first and third quadrants.

Method 6

After discussing the application of *prāṇakalāntara*, the text describes one last method for the determination of *dyujyā* and *prāṇakalāntara*.

> यद्वा द्विनिघ्नीकृतसायनार्कात्
> भुजागुणं कोटिगुणञ्च नीत्वा ।
> परापयानेषुदलाहतौ तौ
> त्रिजीवयाप्तौ भवतः फले द्वे ॥१५॥
> परापमेष्वर्धवियुक्त्रिमौर्व्यां
> कोटीफलं तन्मृगकर्कटाद्योः।
> स्वर्णं च तद्दोःफलवर्गयोगात्
> मूलञ्च भानोर्भवति द्युजीवा ॥१६॥
> भुजाफलं त्रिजीवया समाहतं द्युजीवया ।
> हरेत्फलस्य कार्मुकं कलासुभेद उच्यते ॥१७॥

> *yadvā dvinighnīkṛtasāyanārkāt*
> *bhujāguṇaṃ koṭiguṇañca nītvā* |
> *parāpayāneṣudalāhatau tau*
> *trijīvayāptau bhavataḥ phale dve* ||15||
> *parāpameṣvardhaviyuktrimaurvyāṃ*
> *koṭīphalaṃ tanmṛgakarkaṭādyoḥ* |
> *svarṇaṃ ca taddoḥphalavargayogāt*
> *mūlañca bhānorbhavati dyujīvā* ||16||
> *bhujāphalaṃ trijīvayā samāhataṃ dyujīvayā* |
> *haretphalasya kārmukaṃ kalāsubheda ucyate* ||17||

> Or, after computing Rsine (*bhujāguṇa*) and Rcosine (*koṭiguṇa*) from twice of the precession corrected longitude of the Sun, those two multiplied by half of the Rversine of the maximum declination and divided by the radius (*trijīvā*) become the two *phalas* (*bhujāphala* and *koṭīphala*). The *koṭīphala* is additive or subtractive to the radius diminished by half of the Rversine corresponding to the maximum declination when it is in [six signs] commencing from Capricorn (*mṛga*) or Cancer (*karkaṭa*) [respectively]. The square-root of the sum of the squares of that [result] and *doḥphala* (*bhujāphala*) is the radius of the diurnal circle (*dyujyā*) of the Sun. One should divide the *bhujāphala* which has been multiplied by the radius, by the radius of the diurnal circle (*dyujīvā*). The arc of that result is stated to be the difference in longitude and right ascension (*kalāsubheda*).

This verse first defines two intermediary quantities

$$bhuj\bar{a}phala = \frac{R \sin 2\lambda \times \frac{1}{2} R \operatorname{versin} \epsilon}{R}, \qquad (18)$$

[20]The zodiac signs referred to in the verse serve only to indicate the position on the ecliptic, and are unrelated to the particular star connected to that zodiac sign. Therefore, 'Libra' and 'Aries' here refer to the positions of 0° and 180° on the ecliptic.

$$koṭīphala = \frac{R\cos 2\lambda \times \frac{1}{2}R\,\text{versin}\,\epsilon}{R}, \tag{19}$$

and then gives the following expressions for (i) radius of the diurnal circle

$$R\cos\delta = \left[(R - \frac{1}{2}R\,\text{versin}\,\epsilon \pm koṭīphala)^2 + (bhujāphala)^2\right]^{\frac{1}{2}}, \tag{20}$$

and (ii) the *prāṇakalāntara*, i.e.

$$\lambda - \alpha = R\sin^{-1}\left(\frac{bhujāphala \times R}{R\cos\delta}\right). \tag{21}$$

While it is unclear if the above expressions have any physical significance, one can understand the rationale behind these by comparing with (12)–(15). Whereas (12) and (18) are essentially the same expression, (13) and (19) differ from each other. However, one can easily see that, (14) and (20) yield the same result upon substituting (13) and (19) in them respectively.[21] It appears that the author desired symmetric expressions for *koṭīphala* and *bhujāphala* and suitably modified (19) and (20) to this end.

By substituting (18) in (21), the given expression for *prāṇakalāntara* reduces to

$$\lambda - \alpha = R\sin^{-1}\left(\frac{R\sin\lambda \times R\cos\lambda \times R\,\text{versin}\,\epsilon}{R\cos\delta}\right).$$

The expression within the parentheses is the same as (7), and as we have shown in our discussion there, reduces to $\sin(\lambda - \alpha)$. There, the author directly approximated this expression to the *prāṇakalāntara*. Here, instead, taking the sine inverse of the expression yields the more exact result.

4. DISCUSSION AND CONCLUSION

The *Lagnaprakaraṇa* is an important astronomical work for various reasons. Firstly, as one of the likely works of the great savant Mādhava, the text holds enormous potential for the discovery of new astronomical techniques and insights into the mind of the author, whose works are yet unfortunately poorly studied. Secondly, in contrast to typical astronomical treatises which cover a wide range of topics, the *Lagnaprakaraṇa* focuses exclusively on determining the ascendant, which allows the author to discuss multiple approaches towards solving a given problem. As shown in our discussion, this text reveals the limitless ingenuity of the author in coming up with various expression for *dyujyā* and *prāṇakalāntara*, and showcases him as a true mathematician who delights in solving the same problem in innovative and different ways.

Moreover, these different approaches may help in faster calculations in different circumstances. One can note that the different expressions for the *prāṇakalāntara* discussed in the text involve different trigonometric functions and varying levels of computational complexity. For instance, whereas (1) requires determining the inverse sine only once, (3) requires this twice, adding to computational complexity. However, (3) which is expressed in terms of $\cos\lambda$ rather than $\sin\lambda$, may be more convenient in situations where the value of $\cos\lambda$ is more readily available.

The text also attests to the fact that Indian astronomers were master geometers. For instance, the expressions (5)–(7) have been derived through a deep understanding of spherical geometry as shown, while the expressions (8)–(11) and (12)–(15) suggest that they have been arrived at by conceiving and constructing ingenious epicyclic models, mapping the variation in the radius of the di-

[21] The verse notes that *koṭīphala* is to be applied positively when 2λ is in the range 270° to 90° (six signs starting with Capricorn), and negatively when it is in the range of 90° to 270° (six signs starting with Cancer). This is because *koṭīphala* includes the variable term $\cos 2\lambda$, which is positive in the first and fourth quadrants, and negative in the second and third quadrants.

urnal circle on to the equatorial plane. This is indeed a brilliant strategy. It also appears that the author had a keen eye for mathematical beauty, as seen in the attempts to obtain symmetric expressions for *bhujāphala* and *koṭīphala* in (18)–(19), and was also adept at trigonometric manipulations to arrive at simplified expressions, as seen in the case of (17).

It is clear that a text such as *Lagnaprakaraṇa* would not have been possible without a strong prevalent tradition of mathematical astronomy, and its existence attests to a deep study of the subject in India. The authors hope to throw further light on this important and fascinating work with the publication of the entire text of the *Lagnaprakaraṇa*, along with translation and notes, in the near future.

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