Indian Journal of History of Science, 52.1 (2017) 102-105 DOI: 10.16943/ijhs/2017/v52i1/41306

Book Review

Rao, A B Padmanabha (Trans & Ed.), *Bhāskarācārya's Līlāvatī:* Part I (2016), Part II (2014). Chinmaya International Foundation Shodha Samsthan, Ernakulam, Kerala

M S Sriram*

The immense popularity of *Līlāvatī* and its importance in the history of Indian mathematics is well known. Though *Līlāvatī* is a textbook on arithmetic in popular perception, it has a significant amount of algebra and geometry also. It also improves upon the earlier results on combinatorics. There are many commentaries on it, as also translations and works based on it in the regional languages of India, over the centuries. There is also a Persian translation.

The work was translated into English nearly two centuries back by Colebrooke (*Līlāvatī,* 1927). There have been many other translations, the one by Patwardhan, Naimpally and Singh, being a recent one (*Līlāvatī,* 2001). Now, it has been translated afresh with explanations by A.B. Padmanabha Rao, a mathematician with a distinguished teaching career, and published in two parts. Interestingly, the second part was published first.

He gives the "word (phrase) by word (phrase) translation of the Sanskrit verses in English prose order", so that even a reader without a good Sanskritic background can appreciate what Bhāskara says fully. Many of the explanations are based on the commentary *Buddhivilāsinī* by Gaṇeśa Daivajña, a few are based on *Lilavatīvivaraṇam* of Srīmahīdhara [*Līlāvatī,* 1937], and some on Bhāskara's own explanations in examples. In addition, the author gives his own modern explanations, some of which are insightful. For instance, while discussing the operations with zero, Bhāskara says that if there are further arithmetical operations, a product of a number and zero should not be treated as zero, but zero should be kept beside that number, as a symbol. This is relevant when we have a quantity like $\frac{x.0}{0}$, which is taken to be x by Bhāskara. The author explains this using the concept of 'infinitesimals'.

In *Līlāvatī*, algebraic techniques are used to solve problems in arithmetic and geometry. For example, in *Buddhivilāsinī*, Gaṇeśa uses the solutions of second degree indeterminate equations to generate right angled triangles,where the lengths of all sides are rational numbers. The author draws attention to these techniques.

The organization of the verses in *Līlāvatī* into chapters by the author is some what different from the one in the translation by Colebrooke.

Part I on 'Arithmetic and Algebra' covers verses 1-134, and is divided into 11 chapters. Chapter 1 is on the basic units of measurements of length, area, volume, time, units of money etc., and also conversion from one unit to another. Chapter 2 is on the decimal place value system, and the four arithmetical operations of addition, subtraction, multiplication and division, using various methods. Chapter 3 is on the rest of the four operations of squares, cubes, squareroots, and cuberoots, whereas chapter 4 is on the eight operations on fractions. Chapter 5 is on the eight

* Department of Theoretical Physics, University of Madras, Guindy Campus, Chennai 600025; Email: sriram.physics@gmail.com

operations, when zero is involved. In particular multiplication and division by zero is discussed in great detail. This chapter also describes generation of perfect squares. Quadratic equations are discussed in chapter 6. Chapter 7 deals with the important topic of the law of proportions. Rules of 3, 5, 7, 9 and 11 are explained in detail. The practical applications of the law of proportions to bartered commodities, profit and loss, partnership, purchases and sales, and problems of mixtures are to be found in chapter 8. Chapter 9 is on combinatorics.Bhāskara stresses the importance of this topic in its applications to prosody, architecture, arts, music and other areas. Chapter 10 is on progressions and series. Various progressions, and the sums of corresponding series, partial sums of integers, sum of these sums, sums of squares and cubes are all discussed. Chapter 11 is devoted to arithmetical and geometric progressions, and the sums of the corresponding series.

The broad topics of 'Geometry, First Degree Indeterminate Equations, and Permutations'are covered in Part II, divided into 8 chapters, spanning verses135-272.The first chapter on geometrical figures begins with the theorem of the right triangle, followed by algebraic identities related to it. "Bhāskara's semi-algebraic and Euclid's purely geometric demonstrations of the Pythagoras theorem are compared. The use of algebra in geometry and that of geometry in algebra is illustrated by considering the identities $(a \pm b)^2 \mp 2ab = a^2+b^2$ to demonstrate *Śulbasūtra* (Pythagoras theorem)." Chapter 2 is on the practical applications of this theorem, especially finding the sides separately, given the sum or difference of any two sides, and the third side of a right triangle. These include the famous problems of the 'broken bamboo', 'lotus stem', 'peacock and the snake', and the 'jumping monkeys'. It also includes the projections of the sides on the base in a general triangle, including negative projections. Chapter 3 is on quadrilaterals with special reference to cyclic ones. Bhāskara points out forcefully how a quadrilateral cannot be constructed by specifying just the four sides; one more datum is required which can be a diagonal or a perpendicular from a vertex to a side.There are new results on general and cyclic quadrilaterals. Needle-shaped figures and calculation of their various parts are discussed. Chapter 4 is on the circle. Succesive approximations to π using the principle of doubling the sides of a regular polygon inscribed in it, is described in detail. An approximate relation between the arc and chord of a circle is stated and discussed. In fact this is equivalent to Bhāskara-I's formula for the sine of an angle. The formulae for the area of a circle, surface area and volume of a sphere, and volumes of prisms, cylinders, cones and frustrums are given, with 'proofs' or 'demonstrations' in chapter 5. Chapter 6 on poles and shadows examines the relation among the lamp height, pole height and its distance from the lamp, and lengths of shadows. Chapter 7 is on indeterminate equations of the first degree (*kuṭṭaka*–pulverization problem).Various methods are described to find the solutions. Proofs given in the commentaries *Buddhivilāsini* and *vimala* are explained. The use of *vargaprakṭi* for the generation of integral right triangles in *Buddhivilāsini* is explained. Chapter 8 is on the number of permutations of digits in a number. This includes the case of repeated digits. The special case of a number with a fixed sum of its digits is considered. This forms a part of partition theory in number theory.

In the following, we give two examples of explanations/proofs given in *Buddhivilāsini* which are reproduced in the book under review. In verse 158 of *Līlāvatī*, it is stated that given the sum of the altitude (a) and the base (b) to be s, that is, $a+b=s$, and also the hypotenuse, h, the expressions for a and b are: $a = \frac{s+\sqrt{2h^2-s^2}}{2}$(i) , $b = \frac{s-\sqrt{2h^2-s^2}}{2}$ (ii). The summary of the explanation and a simple extension of the result is given below.

$h^2 = a^2 + b^2$,

$a^2+b^2+2ab=(a+b)^2$, $a^2+b^2-2ab=(a-b)^2$.

$\therefore 2h^2=2(a^2+b^2)=(a+b)^2+(a-b)^2$,

$\therefore 2h^2-(a\pm b)^2=(a\mp b)^2$,

or, $\sqrt{2h^2-(a\pm b)^2}=(a\mp b)$.

From this, we get, $(a-b)=\sqrt{2h^2-(a+b)^2}=\sqrt{2h^2-s^2}$, where $a+b=s$. Solving for a and b, we get (the expressions for them in) (i) and (ii).

Similarly (*Buddhivilāsini* suggests that) we can get $(a+b)=\sqrt{2h^2-(a-b)^2}=\sqrt{2h^2-d^2}$, where $a-b=d$.

Solving for a and b, we get the counter parts of (i) and (ii), namely $a=\frac{d+\sqrt{2h^2-d^2}}{2}$(i) and $b=\frac{-d+\sqrt{2h^2-d^2}}{2}$ (ii).

Another example is the derivation of the expression for the volume of a cone given in the verse 217 of *Līlāvatī*. In the author's own words:

"*Buddhivilāsini* gives a practical demonstration of the formula $V=\frac{1}{3}\pi r^2 h$ (where r is the radius of the base circle of the cone, and h is its height.)"

The volume of a cone is obtained as follows:

Consider a cone of height and base diameter 9 units each. Divide the cone into 9 frustrums each of 1 unit height. The diameters of their bases will be 9, 8, 7, ……, 1 units. ….. the frustrum heights being comparatively small (1 each) and diameters also differ by 1, each frustrum can be treated as a cylinder. In this case, the formula for the volume of the i^{th} frustrum becomes

$V_i=\left(\frac{22}{7}\right)[(r_i+r_{i-1})/2]^2\,1$, ($\pi$ is taken to be 22/7), wherer r_i , r_{i-1}) are the radii of the faces of the i^{th} frustrum.

The sum V, of the volumes V_i, of the frustrums (i goes from 1 to 9), is the approximate volume of the cone. The sum $V=190\,\frac{19}{56}$.

Now the volume of the cylinder is $\left(\frac{22}{7}\right)$ $(9^2/4)\ 9=572\,\frac{11}{14}$. One third of this is $190\,\frac{13}{14}$. which is a very close approximation to$190\,\frac{19}{56}$. "[What the author means is that the result obtained by the explicit summation of the volumes of the frustrums, $190\,\frac{19}{56}$ is a close approximation to the result obtained from the formula stated in the verse, $190\,\frac{13}{14}$. If we divide the cone into 18 frustrums of half-unit high each, the diameters of the intermediate circles would be 9, $8\frac{1}{2}$, 8, $7\frac{1}{2}$,……., 1 and1/2 units each, and the sum of the volumes of the frustrums calculated in the manner above would be $190\ \frac{175}{224}$, which is closer to the actual value, $190\,\frac{13}{14}$.]

The explanation for the expression for the volume of a cone in *Buddhivilāsini* is reminiscent of the derivation of the surface area of a sphere which is given as the product of the circumference and diameter ($=4\pi R^2$) in the *Golādhyāya* part of *Siddhāntaśiromaṇi* of Bhāskara (*Siddhāntasiromaṇi*, 2005). There, he divides each hemisphere into 24 rings, whose radii are $r_i=R\sin\theta_i$, $\theta_i=(\frac{90}{24}\,i)^0$, $i=1, 2,\ldots, 24$. The area of the i^{th} ring is $2\pi r_i\,.\,R\,\delta\theta$, where $R\,\delta\theta$ is the (sloping) thickness of the ring. Bhāskara directly sums the "Rsines", $R\sin\theta_i$ to obtain the area of the sphere. The result he obtains is very close to the product of the circumference and the diameter, and he concludes that the area is actually the product (probably attributing the small difference to the inaccuracies in the values of the Rsines used by him.)

Buddhivilāsini has very many *upapattis* (proofs and demonstrations). The author has included a few of them in this book. Also, the incorporation of the *Buddhivilāsini* explanations are some what uneven in the book. For some demonstrations / proofs, the relevant passages in the commentary in Sanskrit are reproduced, and a reasonable translation is provided. For some others, though the original passages are there, only a summary of the contents of the passages is provided in English, without an actual translation. For many others, even the original is not provided, but only a summary of the explanation is given.This makes the presentation some what uneven. There is a crying need for a full and accurate translation of the *Buddhivilāsini* commentary. However, even though *Buddhivilāsini* is not fully explored in the book under review, it is a very valuable addition to the corpus of literature on *Līlāvatī*, as the author has attempted to present the Bhāskara- Gaṇeśa explanations of the important verses,which gives the reader a feeling for the thinking process of these astronomer-mathematicians of the past, supplemented by modern interpretations also.

Bibliography

Bhāskarācārya's Līlāvatī with Colebrooke's translation and notes by H.C. Banerjee, Second edition, The Book Company, Calcutta, 1927; Reprint, Asia Educational Series, New Delhi,1993.

Līlāvatī of Bhāskarācārya-II, Ed. with *Buddhivilāsini of Gaṇeśa Daivajña and Līlāvativivaraṇa of Mahīdhara* by V.G. Apte, 2 Vols., Pune, 1937.

Līlāvatī of Bhāskarācārya with Translation and comments, K.S. Patwardhan, S.A. Naimpally and S.L. Singh, Motilal Banarsidass, Delhi, 2001.

Siddhāntaśiromaṇi of Bhāskarācārya with his Vāsanābhāśya, Ed. Bapūdeva Śāstri, revised by Gaṇapati Śāstri, Fourth Edition, Chaukhambha Sanskrit Sansthan, Varanasi, 2005.