VERSIFIED SINE TABLES IN JÑĀNARĀJA'S
SIDDHĀNTASUNDARA

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Abstract

The use of sines is an indispensible tool in the computation of planetary position in Indian astronomy, and Sanskrit astronomical treatises include tables of sines or sine differences for this reason. The *Siddhāntasundara* of Jñānarāja, composed around 1500 AD, contains two versified tables, a table of sines and a table of sine differences. In addition to the tables and explanation of their use, Jñānarāja offers a rationale of the first table, which is unique in Indian astronomy. The inclusion of the rationale fits well with Jñānarāja's statement that his treatise is merely the contents of another treatise, the *Brahmasiddhānta*, with demonstrations, *vāsanās*, added.

Key words: India, Indian mathematics, Indian astronomy, Trigonometry in India, History of trigonometry, Sine tables, Jñānarāja, *Siddhāntasundara*.

1. INTRODUCTION

An important but under-studied astronomical treatise in Sanskrit is the *Siddhāntasundara* of Jñānarāja.[1] Composed around 1500 AD, the *Siddhāntasundara* is a comprehensive treatise that covers both the theory behind as well as the practical applications of spherical astronomy.

The *Siddhāntasundara* belongs to the Indian astronomical school known as the *saura-pakṣa*, so called because the astronomical knowledge of the school is believed to have originally been spoken by Sūrya, the Sun God, for the *saura-pakṣa*, see Pingree (1978, pp. 608-618) and Plofker (2009, pp. 71-72). Jñānarāja's treatise was the first comprehensive astronomical work to be written since the composition of the *Siddhāntaśiromaṇi* by Bhāskara II in 1150 AD. One of its prominent characteristics is an attempt to reconcile two very different cosmologies in the

Indian tradition, namely the cosmology of the Indian astronomical tradition and the cosmology of the *purāṇas*, a class of sacred Hindu texts. Jñānarāja's aim was to emphasize the importance and authority of the *purāṇas* within the astronomical tradition. This approach led him to reinterpretation and rejection of certain features of the astronomical model, as well as reinterpretation of passages in the sacred texts, for a discussion of this topic, see Minkowski (2004).

Pingree's *Census of the Exact Sciences in Sanskrit* (Pingree 1970-94) lists over 50 extant manuscripts of the *Siddhāntasundara*, which indicates that the text traveled between scholarly communities and was studied by Indian astronomers. In fact, the text had a significant impact on the development of Indian astronomy in the centuries following Jñānarāja, see Minkowski (2004) for more details.

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2. THE SINE FUNCTION IN THE *SIDDHĀNTASUNDARA*

Jñānarāja commences a theoretical discussion of what we today would call trigonometry at the point in the *Siddhāntasundara* where he proceeds from mean to true planetary motion. Mean planetary motion is discussed in the *madhyamādhikāra* of the *grahagaṇitādhyāya* of the text, while true planetary motion is discussed in the following section, entitled *spaṣṭādhikāra*, for the structure of the *Siddhāntasundara* see Knudsen (2008, pp. 29-35). All references to passages in the *Siddhāntasundara* in this article refer to the edition and translation found in Knudsen (2008), though the translations have been revised.

At the beginning of his discussion of trigonometry, Jñānarāja notes, as other Indian astronomers have noted before him, that more accurate planetary positions are needed in order to, for example, determine the correct timing of the performance of rites, undertaking of journeys, and so on (*Siddhāntasundara* 2.2.1). The key to such increased accuracy, which is achieved by computing the true planetary position from the mean position, is the sine function, an indispensible tool needed to work out the geometrical model of true planetary positions. As such, Jñānarāja opens his section on true planetary motion with a discussion of sines. Sines, including versified tables of values and instructions on how to use the tables, are covered in *Siddhāntasundara* 2.2.2-14. All of these 13 verses are cited, translated, and discussed below.

3. THE SINE FUNCTION IN SANSKRIT TEXTS

In modern mathematics, the trigonometric functions are defined with respect to the unit circle, i.e., a circle of radius 1. In ancient and medieval India, however, the sine function is not defined with respect to a circle of fixed radius, and different authors use different values for the radius of the circle of reference when discussing the sine function (for a recent work on the history of trigonometry from ancient times to about 1550 AD, see Van Brummelen (2009), and for a recent survey of the history of mathematics in India, see Plofker (2009)). In fact, as we shall see is the case with Jñānarāja, an author may introduce more than one value of the radius in the same treatise. Another example is Bhāskara II, who gives tables with radii 3438 and 150 in the *Siddhāntaśiromaṇi*.
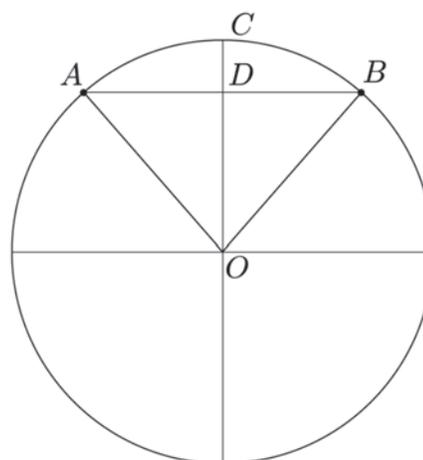

Fig.1: Chord and sine.

In modern mathematics, we generally think of the sine function with respect to an angle. For example, sin(37°) is read as the sine of an angle of 37 degrees. In India, however, a sine was thought of with respect to an arc rather than an angle. For example, in Fig. 1, we think of the segment *AD* as the sine of angle *AOD*, whereas the Indian astronomers thought of the segment as the sine of the arc. The idea is not all that foreign to us since we do essentially the same thing when we work with radians instead of degrees—the measure of an angle in radians is precisely the length of the arc of the unit circle spanned by the angle. We will be using the arc notation below, following the Indian tradition, though we will occasionally make reference to angles.

The sine function in India is derived from the chord function. Since Jñānarāja makes

reference to chords, as we shall see in the following, we note here that in Fig. 1, the chord of the arc *ACB* is the segment *AB*. In modern notation, if α is an angle (or we may think of α as an arc, following the Indian tradition), then *crd* $(2\alpha)=2\sin(\alpha)$, where *crd* denotes the chord function. Since the sine of a given arc is half the chord of the double arc, sines are often referred to in Sanskrit astronomical literature as "half-chords."

A common Indian value for the radius is to have $R = 3438$. This value is used by Āryabhaṭa in the *Āryabhaṭīya*, by Bhāskara I in the *Mahābhāskarīya*, by Bhāskara II in the *Siddhāntaśiromaṇi*, and by other astronomers. However, a number of other values are also used, such as $R = 3270$ by Brahmagupta in the *Brāhmasphumasiddhānta*, $R = 3415$ by Śrīpati in the *Siddhāntaśekhara*, R = 60 by Kamalākara in the *Siddhāntatattvaviveka*, $R = 150$ by Brahmagupta in the *Khaṇakhādyaka*, and R = 120 by Bhāskara II in the *Siddhāntaśiromaṇi* and the *Karaṇakutūhala*. A table of the various values of the radius used by Indian astronomers can be found in Gupta (1978) p. 136.

There is a good reason for the choice of 3438 as the radius. There are 21600 minutes of arc in a circle of 360 degrees, and dividing 2π into 21600′, we get $\frac{21600}{2\pi} = 3437.74677\ldots \approx 3438$. A consequence of this is that if α is measured in minutes of arc and is small, then $3438\sin(\alpha) \approx \alpha$. In other words, using $R = 3438$ is similar to using radians instead of degrees in modern mathematics, where we have that $\sin(x) \approx x$ when x is measured in radians and is small. To give an example with $R = 3438$, we have that $3438\sin(257') = 256.779593\ldots \approx 257$.

It is harder to explain some of the other choices of radius found in the Indian astronomical tradition. However, it may be noted that the choices of R made by the Indian astronomers are often idiosyncratic, perhaps in order to distinguish the material in one work from that found in another, see Gupta (1978) for more details and attempts at rationales for different values of the radius.

Since $R = R \sin(90°)$, the radius R, whatever it may be in the context in question, is referred to as the *sinus totus*, i.e., the "total sine," following Gupta (1978).

4. THE SINE FUNCTION IN THE *SIDDHĀNTASUNDARA*

The *Siddhāntasundara* provides its readers with two versified tables of sines. One is a well-known and standard table that uses a base circle with radius $R = 3438$ and proceeds in steps of 225′ = 3°45′, the other is a table of sine differences that uses the unusual radius $R = 160$ and proceeds in steps of 540′ = 9°. The value $R = 160$ is not found elsewhere in Indian astronomy, but as noted above, the choices of R are often idiosyncratic, and so there is no particular mathematical significance to Jñānarāja's choice of having $R = 160$. For comparison, in addition to the versified table utilizing R = 3438 and proceeding in steps of 225′ = 3°45′ found in the *Siddhāntaśiromaṇi*, Bhāskara II gives another versified table using $R = 150$, which proceeds in steps of 600′ = 10°.

Jñānarāja does not directly specify why he chose to include two versified tables, nor when to use which, but the intention seems to be that the former is the main table of the treatise, while the latter, the entries of which he calls *laghujyās* ("easy sines" or "small sines"), is meant for easy use.

The only place in the *Siddhāntasundara* where we find problems along with sketches of solutions is in the *tripraśnādhikāra* (see, for example, *Siddhāntasundara* 2.3.17). The sketches of solutions given there utilize the *laghujyās*, which indicates that Jñānarāja felt them sufficiently accurate for solving problems. Theoretical discussions, however, generally

assume that the versified table of sines with $R = 3438$ is used.

We will treat the two tables separately in the following.

5. The Versified Table of Sines with $R = 3438$

Jñānarāja's first versified table of Sines is a sequence of 24 numbers (*Siddhāntasundara* 2.2.2-5):

pañcākṛtayo navābdhivedā
bhūśailartumitāḥ khanandanāgāḥ
bāṇābhrabhavāḥ śarenduviśve
viṃśatyakṣabhuvo'ṅkabhūhayakṣmāḥ (2)

āśāṅkabhuvas trigo'bhrapakṣāḥ
saptartvākṛtayaḥ kurāmasiddhāḥ
bhūtebhaśarāśvino 'rdhajīvāḥ
piṇḍarkṣāṇi navākṣapiṇḍasaṅkhyāḥ (3)

vasvadrinavāśvino 'bdhināga-
triṃśac chailanagendurāmasaṅkhyāḥ
aṅgeṣuradāḥ kudasravedāḥ
pakṣādritriguṇāḥ krameṇa jīvāḥ (4)

go'bhrābdhiguṇāḥ kuvahniveda-
lokā maṅgalavahnivedarāmāḥ
āsāṃ tu vilomato 'ntaraikyais
tulyāḥ syur viparītamaurvikākhyāḥ (5)

> The half chords, i.e., the sines, are, in order: two hundred twenty-five, four hundred forty-nine, six hundred seventy-one, eight hundred ninety, eleven hundred and five, thirteen hundred and fifteen, fifteen hundred and twenty, seventeen hundred and nineteen, nineteen hundred and ten, two thousand ninety-three, two thousand two hundred sixty-seven, two thousand four hundred thirty-one, two thousand five hundred eighty-five, two thousand seven hundred twenty-eight, two thousand eight hundred fifty-nine, two thousand nine hundred seventy-eight, three thousand eighty-four, three thousand one hundred seventy-seven, three thousand two hundred fifty-six, three thousand three hundred twenty-one, three thousand three hundred seventy-

two, three thousand four hundred and nine, three thousand four hundred thirty-one, and three thousand four hundred thirty-eight.

The versed sines are equal to the sum of the differences of these taken in the opposite order.

In the above verses, Jñānarāja makes use of the so-called *bhūtasaṃkhyā* system, in which Sanskrit words are used to represent numbers (for the *bhūtasaṃkhyā* system and other Indian numerical systems, see (Sarma 2003)). Since the *Siddhāntasundara* and other Sanskrit astronomical treatises are written in verse, it is necessary to express numbers in words. The *bhūtasaṃkhyā* system is one way of doing that.

Some manuscripts of the *Siddhāntasundara* add numerals into the verses for easy reading. However, whether or not numerals are present in a manuscript depends on the scribe, and as such they have not been included in the above, nor in the other verses cited from the *Siddhāntasundara*.

Note that all that Jñānarāja provides is a sequence of numbers. Other than the statement that the numbers are sines, there is no accompanying explanation of what each number represents or how to use the sequence. A greater context is therefore needed in order to interpret the numbers.

The interpretation of the sequence of numbers is as follows. The versified table proceeds in steps of 225′, i.e., 3°45′, which means that the first number is the sine of 225′=3°45′, the second the sine of 450′=7°30′, the third the sine of 675′=11°15′, and so on, up to the 24th number, which is the sine of 5400′=90°, i.e., the *sinus totus*, the radius $R = 3438$. In other words, the table gives that $R \sin (3°45′) = 225$, $R \sin (7°30′) = 449$, and so on.

It is important to note here that it was most likely Jñānarāja's intention that angular values

should be given in minutes of arc when using the sine table, not degrees (this is not true, however, for the versified sine table using $R = 160$ to be discussed below). However, since we are accustomed to using degrees rather than minutes of arc in modern mathematics, we will make use of both minutes of arc and degrees in the following.

Table 1: The versified table of sines in the *Siddhāntasundara*

Sine number	Arc	Sine	Modern value
1	3°45′	225	224.85
2	7°30′	449	448.74
3	11°15′	671	670.72
4	15°0′	890	889.81
5	18°45′	1105	1105.10
6	22°30′	1315	1315.66
7	26°15′	1520	1520.58
8	30°0′	1719	1719.00
9	33°45′	1910	1910.05
10	37°30′	2093	2092.92
11	41°15′	2267	2266.83
12	45°0′	2431	2431.03
13	48°45′	2585	2584.82
14	52°30′	2728	2727.54
15	56°15′	2859	2858.59
16	60°0′	2978	2977.39
17	63°45′	3084	3083.44
18	67°30′	3177	3176.29
19	71°15′	3256	3255.54
20	75°0′	3321	3320.82
21	78°45′	3372	3371.93
22	82°30′	3409	3408.58
23	86°15′	3431	3430.63
24	90°0′	3438	3438.00

Table 1 lists the sine number, the corresponding arc in degrees, the corresponding sine (the only data given in the *Siddhāntasundara*), and the modern value to two decimal places for comparison.

It is important to note here that the sequence given in the versified table is by no means unique to Jñānarāja. For example, Bhāskara

II gives the same sequence (though the verses giving them are different from the ones of Jñānarāja) in the *Siddhāntaśiromaṇi*.

To study the sequence of sines we will in the following let $\alpha_n = n \cdot 225' = n \cdot 3°45'$ for $n = 0$, 1, 2, ..., 24. For example, $\alpha_0 = 0 = 0°$, $\alpha_1 = 225' = 3°45'$, and $\alpha_2 = 450' = 7°30'$. As such, the nth entry of the table is $R \sin(\alpha_n)$ for $n = 1, 2, ..., 24$. Sine differences will play a role in the following as well, so we let Δ_n denote the nth sine difference, i.e., $\Delta_n = R \sin(\alpha_n) - R \sin(\alpha_{n-1})$ for $n = 1, 2, ..., 24$. To give an example of a sine difference, we have that $\Delta_7 = R \sin(\alpha_7) - R \sin(\alpha_6) = 1520 - 1315 = 205$.

Note that the *Siddhāntasundara* does not provide a ready sequence of sine differences for $R = 3438$, so the reader has to compute them themselves from the values given in the versified table.

6. VERSED SINES

After giving the sines of 3°45′, 7°30′, and so on, Jñānarāja proceeds to note that these can also be used to determine the versed sines of the same arcs. Before proceeding to a discussion of versed sines, it is important to note that there is no tabulation of versed sines in the *Siddhāntasundara*. In other words, the functionality of the versified table of sines goes beyond merely listing sine values—by manipulating the tabulated numbers, we can find the values of other functions, in this case the versed sine function.

The versed sine is a trigonometric function that is no longer in use, but which was utilized by the Indian astronomers in ancient and medieval times. The modern definition of the versed sine of an angle α is $vers(\alpha) = 1 - \cos(\alpha)$. Since $R = 3438$ in Jñānarāja's system, what can be found from the versified table of sines is $R \, vers(\alpha_n) = R - R \cos(\alpha_n)$ for $n = 1, 2, ..., 24$, here expressed in modern notation.

Note that Jñānarāja does not directly speak of the cosine function, and to the Indian astronomers the versed sine is a segment having a certain measure. More specifically, in Fig.1, the versed sine of the arc *AC* is the segment *CD*.

Jñānarāja tells us that the versed sines can be found by adding up sine differences "in the opposite order," i.e., via the formula

$$R \cdot vers(\alpha_n) = \sum_{k=1}^{n} \Delta_{25-k} = \sum_{k=1}^{n} [R\sin(\alpha_{25-k}) - R\sin(\alpha_{24-k})]$$

for $n = 1,2,K,24$. The formula is fairly straightforward and can be proved as follows:

$$
\begin{aligned}
R \cdot vers(\alpha_n) &= R - R\cos(\alpha_n) \\
&= R - R\sin(90^\circ - \alpha_n) \\
&= R\sin(\alpha_{24}) - R\sin(\alpha_{24} - \alpha_n) \\
&= R\sin(\alpha_{24}) - R\sin(\alpha_{24-n}) \\
&= \sum_{k=1}^{n} [R\sin(\alpha_{25-k}) - R\sin(\alpha_{24-k})] \\
&= \sum_{k=1}^{n} \Delta_{25-k}
\end{aligned}
$$

The versified table naturally allows us to find the sines of the arcs α_1, α_2, ..., α_{24}, but it can also be used to approximate the sine of any arc α, with $0 \leq \alpha \leq 90^\circ$, including those not given in the table. We will see below how the table can be used for that purpose.

7. JÑĀNARĀJA'S RATIONALE

Jñānarāja does not merely provide the versified table of sines, but offers a rationale for it as well (*Siddhāntasundara* 2.2.6-9):

vṛtte cakrakalāyute paridhiṣaṇnandāṃśakair aṅkite
prāgrekhobhayacihnayor uparigā jīvā vidheyāḥ kramāt
tāḥ syus tatra gajābdhibhiḥ parimitās tatraiva tāsāṃ dhanūṃsy
ācintyāni dhanur guṇāntaragatā jñeyotkramajyā tathā (6)

ādyaṃ khaṇḍaṃ bhagaṇakalikāṣaṇṇavatyaṃśatulyaṃ
bāhuḥ karṇas tribhavanaguṇaḥ koṭir ābhyām iyaṃ cet
trijyākarṇe bhavati śaradṛṅnetratulye kim evaṃ
labdhaṃ sthūlam prathamaparayor jīvayor antarālām (7)

koṭir digghnī tritithivihṛtā jāyate cāntarālaṃ
tatsaṃyuktāt prathamadalataḥ syād dvitīyaṃ tathaivam
yad yātaiṣyāntarayutidalaṃ jyāntaraṃ tat sphuṭaṃ syād
evaṃ sarvāṇy api guṇadalāny udbhavanti krameṇa (8)

yo mūlamaurvīdalato 'khilāni
jīvādalāny ānayati krameṇa
manyāmahe taṃ gaṇitajñacakra-
bhacakracārākalane dhruvendram (9)

On a circle, consisting of the minutes of arc in a circle and marked with the ninety-sixth parts of the circumference, the chords are to be drawn lying on both of the two points [among the ninety six points marked] that are on a line running east-west.

Accordingly, they are forty-eight. Furthermore, their [corresponding] arcs are to be considered; a versed sine is lying between the arc and [its corresponding] chord.

The first sine is equal to the ninety-sixth part of the minutes of arc in a circle [i.e., two hundred twenty-five]. [It is] the leg [in a right-angled triangle], and the [corresponding] hypotenuse is the radius. If the upright is found from these two when the hypotenuse [in this way] is the radius, what is [the upright] when [the hypotenuse] is equal to two hundred twenty-five? The result thus [obtained] is an approximate [value of] the difference between the first sine and the following one.

[When] the upright is multiplied by ten and divided by one hundred fifty-three, the [sine] difference is produced. [In other words,] the second [sine] is produced from the first sine increased by that [quantity], and so on like that.

Half of the sum of the traversed [sine] difference and the current [sine] difference is the correct sine difference.

In this way, all the sines are produced in order.

We consider the person who computes all half-chords in due order from the first half-chord to be the polestar in the
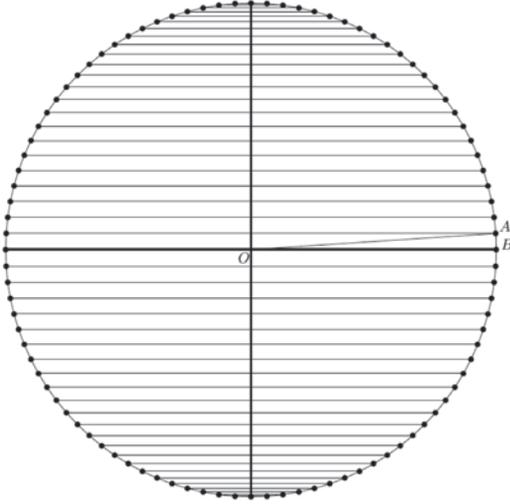
Fig. 2: Construction of forty-eight chords.

fastening of the motion of the circle of those who know computation and the circle of stars.

For the construction of the chords, Jñānarāja considers the circumference of a circle evenly marked with 96 points, as shown on Fig. 2. On the figure, the bold vertical line is the north-south line. Points lying symmetrically about this line are then joined by line segments, running east-west, as shown in the figure (note that to be seen from Jñānarāja's point of view, the figure should be turned degrees). The bold horizontal line is the east-west line. If the two points on the north-south line are counted together one time as a line segment (the reason for this will be clear in the following), the construction gives a total of forty-eight line segments. Note that two consecutive points on the circumference of the circle are separated by $\frac{360°}{96} = 3°45' = 225'$, which is precisely the step in Jñānarāja's versified sine table.

All in all, we see from Fig. 2 that the construction of the 48 line segments gives us forty-eight chords. Note that the two points on the north-south line correspond to the chord of 0° and the chord of 360°, both of which are equal to 0.

If we connect the center of the circle, O, to the point just above the east-west line and to the right of the north-south line, A, and then drop a perpendicular from A to the east-west line, touching it at B, we create a right triangle OAB. The triangle is shown on Fig. 2, though note that the perpendicular is indistinguishable from part of the arc of the circle on the figure.

If the circle has radius $R = 3438$, it follows that the arc corresponding to the segment AB is approximately 225 in length, since the angle AOB is 225′ and $R \sin (\alpha) \approx \alpha$ when α is measured in minutes of arc and is small. As can be readily verified, the length of the perpendicular AB is also approximately 225, i.e., approximately $R\sin (225')$ $= R\sin (\alpha_1)$, in agreement with what Jñānarāja tells us.

The preceding is a standard geometrical derivation of the sines in the Indian tradition, and so it is not unique to Jñānarāja, nor is it of major interest to us. It is in the next step in Jñānarāja's rationale that the material becomes interesting.

As noted by Jñānarāja and seen in Fig. 2, $R \sin (\alpha_1)$ is the leg in a right triangle with hypotenuse R. Jñānarāja next considers a similar but smaller right triangle, though its details are somewhat surprising and its derivation unclear. The meaning of the Sanskrit text is to find a proportion from the similarity of two right triangles: If the upright is found from the leg (i.e., 225) and the hypotenuse (i.e., $R = 3438$) of the original right triangle, what is the leg in the smaller triangle? To answer this, we must know the hypotenuse of the small triangle. The compound word *trijyākarṇe* in the Sanskrit text is somewhat unclear, though the meaning seems to be simply "hypotenuse," as it is taken in the above translation. Separating the compound into the two words *trijyā* (which would mean the radius, i.e., $R = 3438$) and *karṇe* would present the problem of what role the word *trijyā* plays in the sentence. At the same time, if we assume that it is a side in
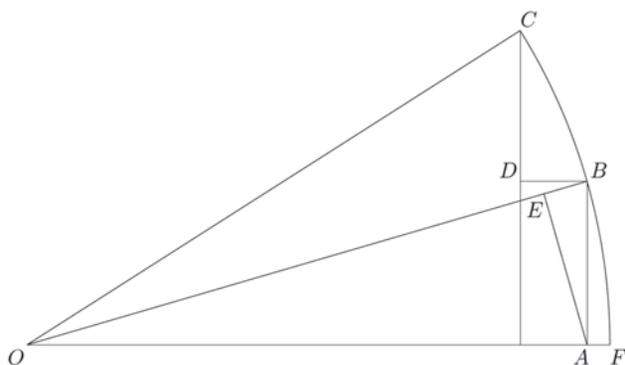
Fig. 3: Two similar triangles.

the small triangle other than the hypotenuse, which is of length 225, the proportion does not yield anything useful.

In other words, the most likely interpretation of the passage is that the hypotenuse in the small triangle is assumed to be 225, i.e., $R\sin(\alpha_1)$. How to visualize the new triangle is not clear, however, and Jñānarāja does not provide any diagrams or instructions on making diagrams to go with his construction. The idea is probably as outlined in the following.

Consider Fig. 3. Assume that the radius of the circle is $R = 3438$ and that the arc FB corresponds to $3°45'$ (note that the figure does not present true proportions since the angle AOB in the figure is not $3°45'$). Assume also that the arc FC is $7°30'$. In other words, the triangle OAB is the large triangle referred to before with hypotenuse $OB = R = 3438$ and leg $AB = R\sin(\alpha_1) = 225$. If we draw the line AE from A perpendicularly onto the segment OB, meeting it at , triangle AEB is the small triangle. It is similar to triangle OAB and its hypotenuse is $AB = 225$.

The upright in the large triangle is OA in Figure 3. From the similarity of the two triangles, we get $\dfrac{OA}{OB} = \dfrac{AE}{AB}$. In other words, we have that

$$AE = \frac{OA \cdot AB}{OB}.$$

Jñānarāja does not consider cosines directly, but if we use modern notation, we get that $OA = \sqrt{R^2 - (R\sin(\alpha_1))^2} = R\cos(\alpha_1)$, and therefore

$$AE = \frac{R\cos(\alpha_1) \cdot R\sin(\alpha_1)}{R} = \frac{221}{3438} \cdot R\cos(\alpha_1).$$

Now, Jñānarāja says that the upright in the smaller triangle, AE, is approximately equal to the second sine difference, $\Delta_2 = R\sin(\alpha_2) - R\sin(\alpha_1)$. In other words, we have that $\dfrac{221}{3438} \cdot R\cos(\alpha_1) \approx R\sin(\alpha_2) - R\sin(\alpha_1) = \Delta_2$. On Figure 3, Δ_2 is the segment DC, so that $AE \approx DC$.

It is not really clear, however, that AE is approximately equal to DC. The following derivation, where we will use the trigonometric relation $\sin(\alpha) - \sin(\beta) = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$ for all α and β, show the relationship between the two quantities:

$$
\begin{aligned}
R\sin(\alpha_2) - R\sin(\alpha_1) &= \frac{2}{R} \cdot R\cos\left(\frac{\alpha_1 + \alpha_2}{2}\right) \cdot R\sin\left(\frac{\alpha_2 - \alpha_1}{2}\right) \\
&= \frac{2}{R} \cdot R\sin\left(\frac{225'}{2}\right) \cdot R\cos\left(225' + \frac{225'}{2}\right) \\
&\approx \frac{2}{R} \cdot \frac{225}{2} \cdot R\cos\left(225'\frac{225'}{2}\right) \\
&= \frac{R\sin(\alpha_1)}{R} \cdot R\cos\left(225' + \frac{225'}{2}\right) \\
&\approx \frac{R\sin(\alpha_1)}{R} \cdot R\cos(225') \\
&= \frac{R\sin(\alpha_1)}{R} \cdot R\cos(\alpha_1)
\end{aligned}
$$

The first approximation follows from the observation that $R\sin(\alpha) \approx \alpha$ if α is measured in minutes of arc and is small.

The weak link in the derivation is the second approximation. We do not know, of course, that the approximation $R\cos(225') \approx R\cos\left(225' + \dfrac{225'}{2}\right)$ was actually used by Jñānarāja. It was certainly not used in the form given by us, since the *Siddhāntasundara* does not make direct reference

to cosines. However, even though Jñānarāja's rationale is unknown to us, using this approximation in the above give us an idea about a derivation, as well as a way to measure its inaccuracy. If we do a direct calculation, we get that $R\cos(225) = 3430.638$ whereas $R\cos\left(225' + \dfrac{225'}{2}\right) = 3421.445$, so to consider that

$R\cos(\alpha_1)$ and $R\cos\left(\alpha_1 + \dfrac{\alpha_1}{2}\right)$ are approximately

equal introduces an error. Just as it is not clear what method Jñānarāja utilized to arrive at his formula, he does not address any issues of error other than telling us that the formula is approximate, thereby acknowledging that there is an error. The error introduced is about 0.27%.

Next, Jñānarāja notes that the $(n+1)^{\text{st}}$ sine difference can be found as $\Delta_{n+1} = R\sin(\alpha_{n+1}) - R\sin(\alpha_n) = \dfrac{10}{153} \cdot R\cos(\alpha \ )$.

From this we get the following relation:

$$
\begin{aligned}
R\sin(\alpha_{n+1}) &= R\sin(\alpha_n) + \frac{10}{153} \cdot R\cos(\alpha_n) \\
&= R\sin(\alpha_n) + \frac{10}{153} \cdot \sqrt{R^2 - (R\sin(\alpha_n))^2}
\end{aligned}
$$

Jñānarāja tells us that this can be used to find each $R\sin(\alpha_n)$ for $n = 1, 2, ..., 24$.

In the formula, the fraction $\dfrac{10}{153}$ is simply an approximation to $\dfrac{R\sin(\alpha_1)}{R}$, since

$$
\frac{R\sin(\alpha_1)}{R} = \frac{225}{3438} = \frac{10}{153} + \frac{5}{58446} \approx \frac{10}{153}.
$$

We can derive the formula for $R\sin(\alpha_{n+1})$ by the same procedure as we just followed in the case $n = 1$. In this case, the derivation becomes:

$$
R\sin(\alpha_1) - R\sin(\alpha_n) \quad = \frac{2}{R} \cdot R\cos\left(\frac{\alpha_{n+1} + \alpha_n}{2}\right) \cdot R\sin\left(\frac{\alpha_{n+1} - \alpha_n}{2}\right)
$$

$$
\begin{aligned}
&= \frac{2}{R} \cdot R\sin\left(\frac{\alpha_{n+1} - \alpha_n}{2}\right) \cdot R\cos\left(\frac{\alpha_{n+1} + \alpha_n}{2}\right) \\
&= \frac{2}{R} \cdot R\sin\left(\frac{\alpha_1}{2}\right) \cdot R\cos\left((2n+1) \cdot \frac{\alpha_1}{2}\right) \\
&= \frac{2}{R} \cdot R\sin\left(\frac{225'}{2}\right) \cdot R\cos\left(n\alpha_1 + \frac{225'}{2}\right) \\
&= \frac{2}{R} \cdot R\sin\left(\frac{225'}{2}\right) \cdot R\cos\left(\alpha_n + \frac{225'}{2}\right) \\
&\approx \frac{2}{R} \cdot \frac{225}{2} \cdot R\cos\left(\alpha_n + \frac{225'}{2}\right) \\
&= \frac{225}{R} \cdot R\cos\left(\alpha_n + \frac{225'}{2}\right) \\
&= \frac{R\sin(\alpha_1)}{R} \cdot R\cos\left(\alpha_n + \frac{\alpha_1}{2}\right) \\
&\approx \frac{R\sin(\alpha_1)}{R} \cdot R\cos(\alpha_n)
\end{aligned}
$$

Similarly to before, it is the approximation $R\cos\left(\alpha_n + \dfrac{\alpha_1}{2}\right) \approx R\cos(\alpha_n)$ that is the weak point of the derivation. Again as before, it is not clear what method Jñānarāja used to deduce the formula. Similarly, there is no discussion of error other than the statement that the formula is approximate.

Before addressing the errors of the Jñānarāja's formula, we address the final part of his rationale (not including the eulogy of those who follow his approach). Jñānarāja's statement that half of the sum of the traversed sine difference and the current sine difference is the corrected sine difference probably means the following. If an arc α satisfies $\alpha_n < \alpha < \alpha_{n+1}$, then the traversed sine difference is Δ_n and the current sine difference is Δ_{n+1}. Consider the following derivation:

$$
\begin{aligned}
\frac{\Delta_{n+1} + \Delta_n}{2} \quad &= \frac{R\sin(\alpha_{n+1}) - R\sin(\alpha_{n-1})}{2} \\
&= \frac{1}{R} \cdot R\cos\left(\frac{2\alpha_n}{2}\right) \cdot R\sin\left(\frac{2\alpha_1}{2}\right) \\
&= \frac{R\sin(\alpha_1)}{R} \cdot R\cos(\alpha_n) \\
&\approx R\sin(\alpha_{n+1}) - R\sin(\alpha_n) \\
&= \Delta_{n+1}
\end{aligned}
$$

From it we get the result stated in the verse. It is not clear, however, why this yields a "corrected" sine difference, nor is it really clear

why we are interested in $\dfrac{\Delta_{n+1} + \Delta_n}{2}$ in the first place. Jñānarāja provides no further comments on this.

8. COMPARISON WITH "PROGRESSIVE CALCULATION"

Jñānarāja's method for computing the sine values of the table differ from the method of "progressive calculation" that Yano (1977, p. 86) notes is employed, either directly or indirectly, by Sanskrit astronomical texts using the sine table R = 3438 with and steps of 3°45′; though Jñānarāja's main table is a table of sines and not sine differences, worth mentioning is also Hayashi (1997), which investigates Āryabhaṭa's rule and table for sine differences. According to the "progressive calculation" method, the $(n+1)^{\text{th}}$ sine can be found by the formula

$$R\sin(\alpha_{n+1}) = R\sin(\alpha_n) + [R\sin(\alpha_n) - R\sin(\alpha_{n-1})] - \frac{R\sin(\alpha_n)}{225}.$$

Even though Yano notes that having $\dfrac{R\sin(\alpha_n)}{233.5}$ as the last term on the right-hand side instead of $\dfrac{R\sin(\alpha_n)}{225}$ would be more accurate, the formula is still much more accurate than the one given by Jñānarāja.

Table 2 gives the sine values as per the table in the *Siddhāntasundara* along with the same values found via Jñānarāja's formula and via the method of "progressive calculation" referred to by Yano. In order to find, say, the 7$^{\text{th}}$ sine via Jñānarāja's formula, the value of the 6$^{\text{th}}$ sine from Jñānarāja's table of sines was used, and similarly for the remaining sine values. For the "progressive calculation," the two previous sine values are taken from the table. It is clearly seen that the values found using Jñānarāja's formula are more inaccurate than those found using "progressive calculation." In fact, given the numbers it appears

Table 2: Comparison with "progressive calculation"

Sine number	Sine	Jñānarāja	"Progressive calculation"
1	225	225.00	225.000
2	449	449.224	448.713
3	671	671.781	671.004
4	890	891.384	890.018
5	1105	1107.046	1105.044
6	1315	1317.783	1315.089
7	1520	1522.619	1519.156
8	1719	1721.551	1718.244
9	1910	1913.601	1910.360
10	2093	2096.838	2092.511
11	2267	2271.267	2266.698
12	2431	2435.933	2430.924
13	2585	2589.893	2584.196
14	2728	2733.145	2727.511
15	2859	2864.753	2858.876
16	2978	2983.800	2977.293
17	3084	3090.284	3083.764
18	3177	3183.311	3176.293
19	3256	3262.880	3255.880
20	3321	3328.141	3320.529
21	3372	3379.12	3371.240
22	3409	3415.818	3408.013
23	3431	3438.124	3430.849
24	3438	3445.331	3437.751

unlikely that Jñānarāja used his own method to generate the sine values. The values are, of course, standard.

9. ARCS BETWEEN 90 AND 360 DEGREES

Jñānarāja's versified sine table covers only arcs between 0° and 90°. However, the sines of arcs between 90° and 360° can also be found via the versified table. How to do so is explained in *Siddhāntasundara* 2.2.10:

> *padāni catvāri sarāśivṛtte*
> *mukhye pade jyopacayaḥ kriyādeḥ*
> *bhaved dvitīye 'pacayas tṛtīye*
> *vṛddhiś caturthe 'pacayo bhuje ca* (10)

There are four quadrants in a circle divided into [twelve] signs [of thirty degrees each]. In the first quadrant, [which spans the first three signs]

beginning with Aries, there is an increase in the sine [when the arc increases]. In the second [quadrant], there is a decrease; in the third, there is an increase; and in the fourth, there is a decrease in the leg [i.e., in the sine].

In Indian astronomy, in addition to being divided into 360 degrees, a circle is furthermore divided into twelve signs of 30 degrees each. These are the twelve zodiacal signs, of which the sign Aries commences at the 0° point of the circle. The quadrants are the same as in modern mathematics.

In modern mathematics, $\sin(\alpha)$ is negative if the angle α is in the third or fourth quadrants, and positive if it is in the first or second quadrants. In the Indian tradition, however, $R\sin(\alpha)$ is always nonnegative. In the verse, Jñānarāja tells us that $\sin(\alpha)$ increases from 0 to 3438 when α increases from 0° to 90°; that it decreases from 3438 to 0 when α increases from 90° to 180°; that it increases from 0 to 3438 when α increases from 180° to 270°; and that it decreases from 3438 to 0 when α increases from 270° to 360°. More specifically,

$$R\sin(\alpha) = \begin{cases} R\sin(180° - \alpha) & 90° < \alpha \leq 180° \\ R\sin(\alpha - 180°) & 180° < \alpha \leq 270° \\ R\sin(360°) & 270° < \alpha \leq 360° \end{cases}$$

As such, if we know $\sin(\alpha)$ for all α with $0° \leq \alpha \leq 90°$, we can find $\sin(\alpha)$ for all α with $0° \leq \alpha \leq 360°$. In the following, however, we will only consider the situation $0° \leq \alpha \leq 90°$.

10. Using the Versified Sine Table

The table as given lists only the sines of $\alpha_1, \alpha_2, ..., \alpha_{24}$, but not the sines of other arcs. In order to compute the sine of an arc α with $0° \leq \alpha \leq 90°$ and $\alpha \neq \alpha_i$ for i = 1,2,...,24 Jñānarāja gives the following instructions (*Siddhāntasundara* 2.2.11):

bhujāṃśakonā navatis tu koṭis
tayoḥ kalāḥ pañcayamāśvibhaktāḥ
phalonmitā jyā vigataiṣyajīvā-
ntaraghnaśeṣāddharalabdhayuktāt (11)

The complement [of a given arc] is ninety [degrees] diminished by the degrees of the arc. The minutes of arc of those two [quantities] are [separately] divided by two hundred twenty-five. The sine whose number corresponds to [the quotient of] each result is increased by what is attained as a result of the division of the divisor [of the previous division, i.e., two hundred twenty-five] into the remainder [of the previous division] multiplied by the difference between the traversed and current sines.

Let α be measured in minutes of arc and satisfy that $\alpha_0 = 0' \leq \alpha \leq 5400' = 90° = \alpha_{24}$. Note that Jñānarāja instructs the reader to simultaneously find both $R\sin(\alpha)$ and $R\sin(90°-\alpha)$, though in the following we will demonstrate only how to find the former; finding the latter is done by the same procedure by replacing α by its complement 90°–α.

Let q and r be, respectively, the quotient and remainder of the division of α by 225. Then $\alpha = 225q + r$ where $0 \leq r \leq 225$. Furthermore, we have that $R\sin(\alpha_q) \leq R\sin(\alpha) < R\sin(\alpha_{q+1})$. Jñānarāja refers to $R\sin(\alpha_q)$ as the traversed sine and to $R\sin(\alpha_{q+1})$ as the current sine. We can now compute $R\sin(\alpha)$ as $R\sin(\alpha) = R\sin(\alpha_q) + \frac{r}{225}(R\sin(\alpha_{q+1}) - R\sin(\alpha_q))$. As already noted, this is common linear interpolation.

To give an example, if we want to compute $R\sin(1440')$, we note that $1440 = 6 \cdot 225 + 90$. Therefore,

$$\begin{aligned} R\sin(1440') &= R\sin(\alpha_6) + \frac{90}{225} \cdot [R\sin(\alpha_7) - R\sin(\alpha_6)] \\ &= 1315 + \frac{90}{225} \cdot (1520 - 1315) \\ &= 1397 \end{aligned}$$

For comparison, the modern value of $R\sin(1440')$ is 1398.360.

Linear interpolation can also be used to find α when $R\sin(\alpha)$ is known (*Siddhāntasundara* 2.2.12):

*jīvāṃ viśodhya śaranetrayamaghnaśeṣā-
dyātaiṣyakhaṇḍavivareṇa hṛtāt phalaṃ yat
yāvat tam ābhavati śodhitamaurvikā tat-
saṅkhyāhatākṣanayanāśviyutaṃ dhanuḥ syāt* (12)

> [Start by] subtracting the [greatest possible] sine [given in the versified table from the given sine]. Whatever is the result of the remainder [of the subtraction of the two sines] divided by the difference between the traversed and current sines, the arc [corresponding to the given sine] is [equal to] that increased by two hundred twenty-five multiplied by the number [corresponding to] how large the subtracted sine was.

Suppose that we are given the value of a sine, for example $R\sin(\alpha)$, and want to find α, which we assume to be between $0°$ and $90°$. First we find the greatest positive integer m that satisfies $R\sin(\alpha_m) \leq R\sin(\alpha) < R\sin(\alpha_{m+1})$. As in the previous, $R\sin(\alpha_m)$ is called the traversed sine and $R\sin(\alpha_{m+1})$ the current sine. We subtract $R\sin(\alpha_m)$ from $R\sin(\alpha)$. The number corresponding to the sine that we subtracted, i.e., $R\sin(\alpha_m)$, is m. According to Jñānarāja, α, measured in minutes of arc, can be found as $\alpha = 225\dfrac{R\sin(\alpha) - R\sin(\alpha_m)}{R\sin(\alpha_{m+1}) - R\sin(\alpha_m)} + 225m$. This, again, is common linear interpolation.

To give an example, suppose that $R\sin(\alpha)$ = 1631. Then we have $R\sin(\alpha_7) < R\sin(\alpha) < R\sin(\alpha_8)$, so that $m=7$. From the formula, we get that

$$\begin{aligned}
\alpha &= 225 \cdot \frac{R\sin(\alpha) - R\sin(\alpha_7)}{R\sin(\alpha_8) - R\sin(\alpha_7)} + 225 \cdot 7 \\
&= 225 \cdot \frac{1631 - 1520}{1719 - 1520} + 1575 \\
&= 1700.502'
\end{aligned}$$

A modern calculation gives us that $R\sin(1700.502') = 1632.117$.

11. The sine table with $R = 160$

As was already noted, for the second versified table, which gives differences of sines rather than sines, Jñānarāja uses $R = 160$ as the radius of the base circle and proceeds in steps of $9°$. The table, like the previous table, is given as a sequence of numbers, 10 in total, in the *Siddhāntasundara* (2.2.13, first three quarters):

*śarākṣiṇī siddhās tridasrāḥ kudasrā
navāṅkā nṛpās trīndavaḥ khendavaś ca
tathāṅgāni rāmā laghujyāntarāṇi* (13)

> The [successive] differences of the easy sines are twenty-five, twenty-four, twenty-three, twenty-one, nineteen, sixteen, thirteen, ten, six, and three.

Similarly to before, we will define $\beta_n = n\cdot 9°$ and $\Delta_n = R\sin(\beta_n) - R\sin(\beta_{n-1})$ for $n=1,2,....,10$. The interpretation of the sequence of numbers is that each number is not directly a sine, but a sine difference. For example, the fourth number, 21, is the difference $\Delta_4 = R\sin(\beta_4) - R\sin(\beta_3)$. We can therefore reconstruct the actual sine values as $R\sin(\beta_n) = \sum_{k=1}^{n} \Delta_k$. Jñānarāja calls the tabulated sine differences *laghujyā* ("easy sines" or "small sines").

Table 3 lists the sine number, the corresponding arc, the sine difference Δ_n (the only data given in the *Siddhāntasundara*), the corresponding sine, the modern value of the sine, and the modern value of the sine difference (the latter two for comparison).

There is no doubt that Jñānarāja's reason for including the second sine table is that it is easier to do computations with the sine differences given therein; the numbers are smaller and there are fewer of them. However, the increase in ease of use comes at a cost, namely that of less accuracy, due to the step between sine differences being larger than the step between sines in the previous table.

Table 3: The versified table of sine differences in the *Siddhāntasundara*

Sine number	Arc	Δ_n	$\sin(\beta_n)$	Modern value	Modern difference
1	9	25	25	25.029	25.029
2	18	24	49	49.442	24.413
3	27	23	72	72.638	23.195
4	36	21	93	94.045	21.407
5	45	19	112	113.137	19.091
6	54	16	128	12.442	16.305
7	63	13	141	142.561	13.118
8	72	10	151	152.169	9.607
9	81	6	157	158.030	5.861
10	90	3	160	160.000	1.969

As with the previous sine table, the present table 3 can be used to compute the sine of a given arc β, as well as the arc corresponding to a given sine $R \sin(\beta)$ (*Siddhāntasundara* 2.2.13 (last quarter) and 2.2.14 (first two quarters)):

bhujāṃśānavāptā gatajyātha śeṣam (13)

*ayātāhataṃ gohṛtaṃ yātakhaṇḍair
yutaṃ jyā bhaved atra khaṇḍāṇi jahyāt* (14)

The degrees of the arc are divided by nine. [The quotient of the division] is [the number of] the traversed sine. The remainder [of the division] is multiplied by the current sine and divided by nine. [The result is then] increased by the traversed [sine] differences.

Note that the procedure given here by Jñānarāja relies on β being measured in degrees, not in minutes of arc as was the case with the versified table using R = 3438.

Let β be an arc and let q and r be the quotient and the remainder, respectively, of the division of β by 9. Then β = 9q + r, where $0 \le r < 9$. The sine of β can now be found as

$$R\sin(\beta) = \frac{r}{9}\Delta_{q+1} + \sum_{k=1}^{q}\Delta_k.$$ This is, again, linear interpolation.

As an example, if β = 71°, then 71 = 9·7 + 8, and therefore

$$\begin{aligned} R\sin(\beta) &= \frac{8}{9}\cdot\Delta_8 + (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6 + \Delta_7) \\ &= \frac{8}{9}\cdot 10 + (25 + 24 + 23 + 21 + 19 + 16 + 13) \\ &= 8.889 + 141 \\ &= 149.889 \end{aligned}$$

The modern value of $R \sin(71°) = 160 \sin(71°)$ is 151.283.

If instead of an arc we are given the value of a sine, say $R \sin(\beta)$, Jñānarāja gives us a rule for finding β as follows (*Siddhāntasundara* 2.2.14 (last two quarters)):

*navaghnāvaśeṣād ayātāptalabdhaṃ
viśuddhāṅkanighnāṅkayuktaṃ dhanuḥ syāt* (14)

One should subtract [as many of] the [sine] differences [from the given sin as possible]. [When] the result from the division of the remainder [of the subtraction process] by the current [sine difference] is increased by nine multiplied by the number corresponding to the traversed [sine difference], [the result] is the arc.

Let n be the largest possible integer that $r = R\sin(\beta) - \sum_{k=1}^{n}\Delta_k$ satisfies that is nonnegative. Put in another way, we subtract as many successive sine differences as possible from $R \sin(\beta)$, yielding the result r. Then β can be found as $\beta = \frac{9r}{\Delta_{n+1}} + 9n$.

To give an example, if $R\sin(\beta) = 77$, we can subtract three sine differences, so that n = 3 and r = 77 – (25 + 24 + 23) = 5. We then find β, measured in degrees, as

$$\beta = \frac{9 \cdot 5}{\Delta_4} + 9 \cdot 3 = \frac{45}{21} + 27 = 29.143$$

The modern value is $\arcsin\left(\frac{77}{160}\right) = 28.767°$.

This is, once again, linear interpolation.

12. Conclusion

On the whole, Jñānarāja's treatment of sine tables and sines in the *Siddhāntasundara* remains broadly within what one would expect to find in a Sanskrit astronomical treatise. For example, the main versified sine table, which uses the radius $R = 3438$, is identical to the one given by previous astronomers, such as Bhāskara II. While the radius $R = 160$ of Jñānarāja's versified table of "easy sines" is not found elsewhere in Indian astronomy, which is true also for the step of 9° used for it, the creation of such "easy" tables is idiosyncratic and as such it does not stand out. Also the fact that Jñānarāja gives two tables in his section on sines remains within the tradition. For example, as was noted above, in addition to the main table with radius $R = 3438$, Brahmagupta uses the radius $R = 150$ with steps of 15°. Finally it may be noted that Jñānarāja's presentation of one table of sines and another of sine differences is also within the tradition, as both are found among the writings of his predecessors in the Indian astronomical tradition. In other words, only with respect to the radius and step of the table of "easy sines" does Jñānarāja show originality.

The point where Jñānarāja's material does stand out, however, is in his rationale for generating the sine values of the versified table with $R = 3438$. Rather than merely presenting a versified table of sines, Jñānarāja decided to include a procedure for computing the sines one by one, starting with the first sine, $R \sin (3°45') = 225$.

That Jñānarāja includes such a procedure is itself not surprising. In fact, it is consistent with a statement (*Siddhāntasundara* 1.1.3) of his that the *Siddhāntāsundara* presents the astronomical knowledge of the *Brahmasiddhānta* to which is added *vāsanās*, i.e., demonstrations. (loosely, proofs, though we need to be aware that the Indian mathematical tradition did not operate with proofs in the Euclidean sense). The *Brahmasiddhānta* is a somewhat obscure astronomical treatise, dated by Pingree (1981, p. 26) to the 350-year period between the composition of the *Siddhāntaśiromaṇi* by Bhāskara II in 1150 CE and the composition of the *Siddhāntāsundara* by Jñānarāja (for a critical edition of the *Brahmasiddhānta* with an introduction in English, see Dhavale (1996). According to the tradition it presents a lecture given by the god Brahmā to the sage Nārada and recorded by the sage Śakalya. Jñānarāja's attempts to supply the *Brahmasiddhānta* with *vāsanās* is part of his agenda to emphasize the importance and authority of the *purāṇas*; *vāsanās* belong to the prestigious tradition of Bhāskara II, and an addition of them to the *Brahmasiddhānta* would increase the text's importance making a stronger case for respecting the authority of the *purāGas* in Indian astronomical tradition (for a discussion of *vāsanās* and the *Brahmasiddhānta*, see Knudsen (2008, pp. 4-6, 35-41).

In modern terms, Jñānarāja's procedure amounts to

$$R\sin(\alpha_{n+1}) \approx R\sin(\alpha_n) + \frac{10}{153}R\cos(\alpha_n)$$

It differs from other procedures, such as the "progressive calculation" noted by Yano and discussed above, and is unique in the Indian tradition. Jñānarāja's method, however, is not all that accurate and Table 2 shows that it produces numbers that are somewhat off the mark, and probably unacceptably so.

The brief nature of the verses in the *Siddhāntasundara* (in fact, of verses in Sanskrit astronomical literature in general) complicates the reading and understanding of the procedure. The geometrical constructions alluded to by Jñānarāja, but not explained clearly in words or via diagrams, are hard to visualize. In other words, it is not fully clear how Jñānarāja arrived at the procedure, nor is it clear why he departs from previous procedures. Perhaps the procedure was done hastily by Jñānarāja in order to procedure a *vāsanā* for his main sine table. At any rate, the appearance of a procedure in the discussion of sines fits perfectly with Jñānarāja's stated aim of presenting a text that presents the astronomical system of the *Brahmasiddhānta* supplemented with *vāsanās*.

NOTES

1. The most comprehensive introduction to the *Siddhāntasundara* is currently (Knudsen 2008), which includes a critical edition, English translation, and commentary on selected chapters of the text. An edition, though not critical, of the entire text of the *Siddhāntasundara* can be found in (Śukla 2008).

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