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TECHNIQUES OF ANCIENT EMPIRICAL MATHEMATICS

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The paper deals with techniques used in primitive and ancient mathematics. For the length of the arc of a circular segment, a newly discovered old Babylonian rule and an ancient Indian formula are discussed. For obtaining the approximations and limits of square roots, the quite simple method of squaring and cubing has been described. Equivalence with other usual methods has been shown. The ancient popular process of averaging for computing areas and volumes is illustrated with several examples. The simple Golden Rule of Three (*trairāśika*) has been dealt in quite wider sense with a indent variety of uses in history of mathematics.

The analogy principle as a method of proof was a common and powerful tool in empirical mathematics. Its use in a wide range of mathematical formulas has been discussed. Interpretations of Āryabhaṭa I's empirical formulas for the volume of a tetrahedron (*ṣaḍaśri*) and sphere have been freshly presented from ancient sources. Representation of mathematical quantities through ancient popular unit fractions have been dealt. Some miscellaneous topics such as computation of tabular Sines and the process of iteration (*asakṛta-karma*) have been included. The paper is fully documented.

Key words: Ancient and Medieval Mathematics; Āryabhaṭa I; Empirical and Primitive Mathematics; Method of Analogy; Process of Averaging; Pythagoras Theorem; Rule of Three; Segment of a Circle; Square Roots; Unit Fractions; Vedic Mathematics.

1. INTRODUCTION

The subject of mathematics can be said to have great antiquity. Among the three famous old topics of reading, writing, and arithmetic, the last one is the most ancient because the other two needed some sort of script to be developed.

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The idea of whole number and the process of counting are very ancient. Artifacts of numerical significance (e.g. those containing notches for counting) which are older than twenty thousand years, have been found[1]. Thousands of tokens (small clay objects of varied shapes) were produced in the middle east during the period 8000 to 6000 BC. After careful examination of the tokens, Denise Schmandt-Besserat[2] had concluded that they were used for concrete counting entailing both cardinality and object specificity. In fact "to calculate" earlier meant "to reckon by means of pebbles" (the word "calculus" comes from the Latin *calx* which means "stone").

Ancient Babylonian, Chinese, Egyptian, and Indian mathematics were all concerned with rectilinear as well as curvilinear measurement. But their treatment could not be granted full-fledged mathematical stature for want of deductive reasoning (from first principles) and rigorous logical procedure. Also, frequently there was no clear cut distinction between results which were exact and those which were approximate only.

Another lacuna in primitive mathematics was that proofs of formulas and other mathematical relationships were not explicitly brought out through deductive logic. The main reason was that mathematics was not studied commonly for its own sake. During antique remote times, this situation was usual in most of the cultural areas of the world.

In India of Vedic period, mathematics as well as astronomy were studied and developed for religious purpose. Thus it seems that the aim of early Indians was not to build up an edifice of logically deductive science of mathematics on the foundation of a few self evident fundamental axioms (as was done in Greek mathematics later). Even a visual demonstration or a non-rigorous derivation and explanation was quite an accepted from of the proof.[3] Moreover, empirical reasoning was often considered sufficient. Also, generally the proofs (whatever sort they might have been) were supposed to be explained orally by the teachers to students. Frequently it was left to the commentaries to give exposition by including possible derivations or rationales and other details.

When proofs of the theorems and formulas are found in later sources, the tools used in them should be examined. If they are attributed to the older sources, the availability of the said methods in older times should be checked. It should also be noted that if simpler or empirical techniques enable us to get the needed rules and results, we should be careful in assigning more sophisticated or more general methods of later period to old sources.[4]

2. Some Simple Empirical Rules

In a given circular segment (Fig.1), let the length of the chord PQ be c and the height of the segment be h (=EM). A modern exact method of finding the segment's arc-length PEQ (= s) is to make use of the trigonometrical formula

$$s = d \sin^{-1} (c/d) \qquad \ldots(1)$$

where the diameter d (= 2r) of the circle is given by the rule

$$C^2 = 4h (d - h) \qquad \ldots(2)$$

But more than 350 years ago (when such trigonometrical method was not known), the Babylonians made use of the empirical relation

$$s = c + h \qquad \ldots(3)$$

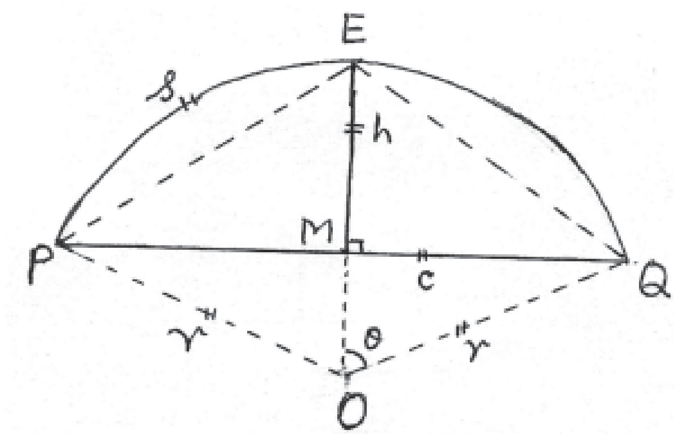

Fig. 1.

The supposed use of this simple formula is based on certain calculations found in the old Babylonian text BM 85194 which is dated about 1600 BC[5]. The details of the discovery of the pre-trigonometry empirical formula (3) are given in a recent paper[6]. A simple and possible empirical way of arriving at the formula is also suggested in the paper as follows. Consider various segmental arcs on the chord PQ (Fig. 2). When the height h is zero, the curved arc PEQ coincides with the straight chord PQ. As the curved arc moves more and more away from the

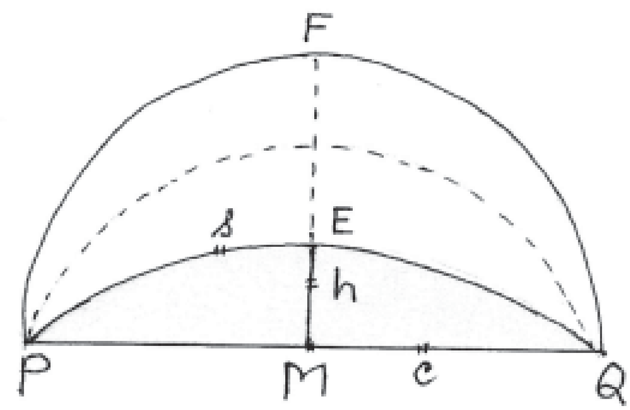

Fig. 2

chord PQ, the excess of the length of arc PEQ are PQ as well as the height h both increase. That is, (s-c) increases with h. Assuming a simple proportionate variation, we have

$$s - c = \lambda h \qquad \ldots(4)$$

where λ is the linear proportionality constant. This constant can be easily found by taking the simple case of semicircle (which is also a segment) on PQ as diameter. That is,

when $c = 2r$, $s = \pi r$, $h = r$.

Putting these in (4) we get $\lambda = 1$ for the then commonly used simple Babylonian value $\pi = 3$. Hence we get (3). Another empirical derivation of (3) follows if the segmental arc PEQ (Fig. 1) to treated analogous to a semicircle for which (3) is true with

$\pi = 3$, so that $s = 3r = 2r + r = PQ + ME = c + h$.

About 2000 years later and about 2000 miles east of Babylonia, we come across a different type of rectification of the circular segment. The new empirical formula is found in the Jaina School in India. It can be expressed as[7]

$$s = \sqrt{c^2 + kh^2} \qquad \ldots(5)$$

where

$$k = \pi^2 - 4 \qquad \ldots(6)$$

The derivation seems to follow the reasoning thus (Fig. 1):

s = arc PEQ = arc PE + arc EQ

>(chord PE + chord EQ)

$$= 2PE = 2\sqrt{\left(c/2\right)^2 + h^2}$$

That is, …(7)

So that s can be assigned the form (5) empirically provided that k is greater than 4. Finally, to find k numerically, the case of semi-circle (which is also a circular segment) was considered there by getting (6) by putting c = 2r, h = r, and s = πr in (5).

For the commonly used Jaina valued $\pi = \sqrt{10}$ in (6), the value of k will be 6, and (5) becomes

…(8)

$$s \geq \sqrt{c^2 + 4h^2} \quad s = \sqrt{c^2 + 6h^2} \quad \sqrt{10}\,r = \sqrt{(2r)^2 + 6r^2}$$

This formula (8) is found in most of the Jaina works on mathematics and cosmography (in prakrit and sanskrit) including those of Umasvāti who is variously placed from 1st to 5th century AD.[8] Interestingly, if we replace c^2 in (8) from the well known formula (2) and simplify, we get the form

$$s = \sqrt{2\left[\left(d + h\right)^2 - d\right]} \qquad \ldots(9)$$

This peculiar form is found in the *Tiloya-paṇṇatti*, IV. 181 of Yativṛṣabha[9]. Mahāvīra in his *Gaṇitasāra-saṅgraha* gives (8) as well as the case for k = 5 (which corresponds to π = 3) while the form corresponding to π = 22/7 is found in the *Mahāsiddhānta* of Āryabhaṭa II who was not a Jain.[10]

As already pointed out earlier, the analogy between segment and semicircle can also be used to derive the rectification formulas by empirical generalization. For example, for the semicircle of radius r, the curved arc is s =   r, using the usual Jaina value of π. Now we write it as

…(10)

With respect to semicircle (Fig. 2), 2r in (10) is the base chord PQ and r is the height MF. So, by replacing 2r by c and r by h in (10) we get, of course analogously, the empirical rule (8) for segment PEQ (Fig. 1). This primitive type of analogy is quite crude. However, analogy in a wider sense was an accepted method of proof (see the next section).

3. Analogy and Some Rules of Āryabhaṭa I

Interpreted in the wider sense of similarity, the analogy as method of proof has been quite common in mathematical sciences since long in all cultures. It seems to be based on the general belief that the world itself was a mathematical creation in which all things were connected by a common mathematical plan. Many eminent mathematicians such as Āryabhaṭa, Kepler, Newton, and Euler, relied heavily on analogical reasoning. Often new discoveries are made on the basis of analogy and their justification and demonstration are found later on.

When Newton extended the binomial theorem over to negative and fractional index, he appealed to the uniformity of nature which is a sort of analogy principle. Euler often used analogy to extend mathematical notions and concepts. He defined "sum" of any series, the expression whose expansion yields that series (even if it is not convergent). For instance, if we divide 1 by (1-x) by the usual method, we get

$$1/(1-x) = 1 + x + x^2 + x^3 + \dots\dots\dots\dots$$

So he took 1/(1-x) as the 'sum' of the RHS series for all finite values of x, e.g. he got the absurd result

$$1 + 2 + 4 + 8 + \dots\dots\dots = -1$$

by putting x = 2 in the above series! Following such generalized concept, Ramanujan got[11]

$$1 - 2 + 3 - 4 + 5 - 6\dots\dots\dots = 1/4$$

from the expansion of $1/(1+x)^2$ and then putting x = 1.

In primitive times, the analogical principle was helplessly employed when exact or better method seemed to be out of reach. The so-called Jorge's formula[12].

$$\text{Area, } A = (p/4)^2 \qquad \dots(11)$$

for the area of a general quadrangular field with sides a,b,c,d, was a choice in primitive lines. Here

$$p = (a + b + c + d) \qquad \ldots(12)$$

is the perimeter of the field. The formula (11) was based on the analogy of the area of a square for which it is true. An equivalent of (11) is also found in the Latin work of Alcuin (about 800 AD)[13]. To estimate area through perimeter is an older practice although (11) may be seen to imply the concept of average also.

According to Boyer[14], the Rhind papyrus (c. 1650 BC) shows shat the Egyptians correctly found the volume of a square pyramid to be one-third the volume of the right prism having the same base and altitude. According to the Archimedes' treatise Methods, the formula

Vol. of pyramid = (1/3) (vol. of prism) …(13)

(the polygonal base and height being same) was known to Democritus (c. 400 BC) who also knew that a similar relation exists between the cone and the cylinder.[15] But it is surprising to find that Maimonides (1135-1204) in his *Moreh N Vokheem* speaks of those who still thought the cone to be half of the cylinder with the same base and height.[16]

In India, Āryabhaṭa I (born 476 AD) states in his *Āryabhaṭīya* II. 6 (2nd half) that the volume of a tetrahedron (*ṣaḍaśri* or six-edged solid) is half the product of the area of its triangular base and height.

That is, for a regular pyramid with triangular base

Vol. V = (1/2) (area of base).(height) …(14)

which is wrong. The correct formula (13) is found in the *Brāham-sphuṭa Siddhānta* XII. 44 of Brahmagupta (628 AD). However, the surprising thing is that most of the commentators of the *Āryabhaṭīya* made no fuss about (14) even as late as Kodaṇḍarāma (c.1850).[17]

Many explanations have been suggested for Āryabhaṭa's mistake or confusion. One of them is that (14) is based on the speculation of analogy with the formula for the area of a triangle (dealt in the 1st half of II. 6) namely

Area, A = (1/2) (length of base).(height) …(15)

It is interesting to note that the Greeks associated the metaphysical element fire with a tetrahedron while in India the triangle with apex upwards was also called *agni* (fire) or Śiva triangle.[18] Another explanation is that, for the volume of a frustum of a pyramid, an ancient formula (based on the frequently used habit of averaging) could be

$$\text{Vol.} = (1/2)\,(A + A').(\text{height}) \qquad \ldots(16)$$

where A and A′ are the areas of the base and top.[19] In the case of a pyramid, A′ = O, and we get (14).

Some very artificial and twisted interpretations of the Āryabhaṭa's rules have been also given [20] but they are not supported by texts or commentators. An important point to note is that the falsehood of (14) could have been easily found by making some models or even by weighing crude replicas.

There is also a mathematically important point to note. Āryabhaṭa's mistaken formula (14) is not in harmony with his correct rule for the total number of small balls or shots which form a triangular pyramidical pile. Counted from the top, the n^{th} layer will have

$$1 + 2 + 3 + \ldots\ldots + n = n(n+1)/2 \text{ balls} \qquad \ldots(17)$$

According to the *Āryabhaṭīya* II. 21, the total number of balls in the n layers will be given by

$$n(n+1)(n+2)/6$$

which is the *citighana* or voluminous contents of pile. In the same spirit (17) will represent the oral contents of the n^{th} layer or the triangular base, and n (the number of layers) the height. So by assuming (for the triangular pyramid)

$$\text{Volume} = k(\text{area of base}) \times (\text{height}) \qquad \ldots(18)$$

We should have, roughly speaking, in the limit (as $n \to \infty$)

$$n(n+1)(n+2)/6 = k.[n(n+1)/2].n$$

By taking the limits in this, we easily get k=1/3 using which in (18), will lead to the correct formula(13)*.

* For a pile in the shape of a square pyramid (with n^2 ball in the nth layer), the next verse (II.22) gives the voluminous contents as n(n+1)(2n+1)/6. So here also $k = \lim_{n\to\infty} n(n+1)(2n+1)/(6n^2.n) = 1/3$ correctly.

For the mensuration of a circle and sphere, certain bold analogies have been used in the history of mathematics. The Jorge's formula (11) used analogously for a circle of circumference C, will give

area of circle = $(C/4)^2$ …(19)

This was applied not only in primitive mathematics[21] but was suggested even in 1894 by Goodwin in America.[22] Interestingly, attempts to legalise (19) were made in the U.S.A. through the notorius house Bill No. 246 (Indiana State Legislature, 1897) but they could not succeed.

The *Āryabhaṭīya* II. 7 (1st half) contains correctly a rule for the area of a circle equivalent to

Area = $(C/2)\cdot(D/2) = C.D/4$ …(20)

where the width (*viṣkaṃbha*) or diameter $D = 2r$.

The rule (20) is very ancient and quite common. Interestingly, it is true for the square ($C = p = 4a$, and $D = a$ = side of square) and was used for general round plane figures.

A similar analogy exists between cube (of side a) and sphere (of radius r) for their volumes in respect to the formula.

Volume = (total surface) × (width)/6 …(21)

which yields the exact volume in each case.

For cube, total surface = $6a^2$, width = a, and by (21)

Vol. $V_1 = (6a^2).\ a/6 = a^3$, correctly …(22)

For sphere, surface = $4\pi r^2$, width = 2r, and by (21),

Vol. $V_2 = (4\pi r^2),\ 2r/6 = (4/3)\ \pi r^3$, correctly …(23)

Now for the cubic volume, the formula (22) can also be written as

$V_1 = (a^2).a$

$= M\sqrt{m}$ …(24)

where M is the area of the middle section which passes through the centre of the cube and lies half way between a pair of opposite faces. Āryabhaṭa I seems to have followed the rule (24) analogously for the sphere.

In *Āryabhaṭīya* II.7 (2^{nd} half) he says

tannija-mūlena hataṃ ghana-gola-phalaṃ niravaśeṣam

"That (i.e. the area of a circle mentioned in the first half of the verse) multiplied by its own (square) root is the volume of a sphere (whose central section is the above circle) without remainder (i.e. exactly)."

That is, the volume of a sphere of radius r is

$$V = A\sqrt{A} \qquad \ldots(25)$$

where A is the area of the central (or greatest) circular section and which, by (20), is given by

$$A = (2\pi r).(2r)/4 = \pi r^2 \qquad \ldots(26)$$

Putting this in (25) we have the wrong formula

$$V = \left(\pi r^2\right)\sqrt{\pi r^2} \qquad \ldots(27)$$

Thus we see that the analogy of cube and sphere works alright for (21), but fails for (24). $\sqrt{\pi r^2}$

Of course, the correct volume of a solid can be found by applying a more general rule

Vol. = (chosen sectional area) × (effective height)

provided the effective height is properly found out. For the central section of a sphere the correct effective height (*ucchrāyaḥ*, as Parameśvara call it)[23] is 4r/3 and not (= 1.77 r nearly) as implied in (27).

Correct volume of a sphere (or any solid) can also be found by determining the side of a cube (called *dvādaśāśra* by Nīlakaṇṭha) of equal volume. But this effective side is not equal to the side of a square (*caturaśra*) whose area is equal to the central section of the sphere. That is, although square of side ($\sqrt{\pi}.r$) will give area equal to that of a circle of radius r, the cube on side ($\sqrt{\pi}.r$) will not yield the volume of the sphere of radius r. Thus the analogy pointed out by Nīlakaṇṭha[24] as an explanation of Āryabhaṭa's rule (27) does not work. The correct effective side of a cube equal in volume to the sphere will be, using (23),

$$\text{side, } s = (4\pi/3)^{1/3}.r = 1.61 \text{ r nearly} \qquad \ldots(28)$$

while Āryabhaṭa's rule (27) implies = 1.77r, nearly. The error here is less than in treating $\sqrt{\pi}r$) as effective height.

Thus we find that Nīlakaṇṭha's interpretation of Āryabhaṭa's rule is far better than that of Parameśvara. Correct formula for the volume of a sphere was known to Archimedes (c. 225 BC) and many empirical rules were also known.[25] Āryabhaṭa called his rule as exact (*niravaśeṣam*). He showed his originality in his attempt to find correct volume through effective side or height.

4. Square-roots by Squaring and Cubing

The extraction of square-roots is a frequently employed operation in computation beyond rational arithmetical operations. The square root of a non-square positive integer N is irrational and so its true or exact numerical volume cannot be expressed as a terminating decimal or represented by a fraction p/q of two integers. But the value of $\sqrt{N}$ can be found approximately or to any desired degree of accuracy by some simple methods. One of the earlier such method is the process of squaring and cubing.

The practical working of the method may be illustrated by taking an example. Here we take the evaluation of $\sqrt{10}$ which is called the Jaina value of π in historical context. For calculating the circumference of a circle of diameter D, the Jaina School commonly used the empirical rule

$$C = \sqrt{10D^2}$$

$$(\sqrt{10} - 3) = 1/(\sqrt{10} + 3) < 1/(3+3) = 1/6$$

…(29)

which may be taken to imply the use of $\sqrt{10}$ for π. However, it must be noted that their actual numerical computation of C was based on the ancient empirical formula

$$\sqrt{a^2 + x} = a + (x/2a) \qquad \text{…(30)}$$

Rather than on the direct multiplication of $\sqrt{10}$ and D.[26]

Now the square number nearest to 10 is 9 and

Thus the error e in taking $\sqrt{10} = 3$ is less than one numerically. So that

$$e^n = (\sqrt{10} - 3)^n, \quad n = 1,2,3 \qquad \text{…(31)}$$

will form a decreasing sequence (converging to zero). In this way by expanding the right hand side of (31) for n =2, 3, 4 etc. and equating to zero each time, we can easily find improvingly better and rational approximations for √10. For n = 2, we have

…(32)

which gives = 19/6. This value, which also follows from (30) with a = 3 and x = 1, is often found separately among the Jainas and elsewhere. For n = 3 and 4, we have

Expanding and simplifying we get

From these we get the approximations [illegible]

= 117/37, and (the better one) = 721/228 …(33)

Decimally 721/228 is 3.162281 nearly while the correct value is 3.1622776 nearly (correct to 7 decimals).

It may be pointed that the approximation 721/228 was also obtained by Rhabdas (c. 1340) as well as by his Indian contemporary Nārāyaṇa Paṇḍita but they followed different methods.[27] Nārāyaṇa also obtained the still better approximation 27379/8658. Here this value can be easily obtained by zeroing square of e^3 (which is already found above) i.e. by

(to be squared first).

The above elementary method of squaring and cubing can also be used to find fine limits between which the value of a simple surd lies. To illustrate this we take a historically famous example. It is known that the famous Greek scholar Archimedes (c. 225 BC) stated

…(34)

Several explanations are available for the limits set here and they have offered a great fascination as well as challenge to historians of science to reach the original derivation of Archimedes.[28] The following simple method was given first by T.N. Thiele [29] as early as 1884 but is not found in standard histories of mathematics. The square of 5/3 is 25/9 which is less than 3, so that 5/3 is less than √3 and we have (5-3√2) < 0.

Thus by expanding $(5-3\sqrt{3})^n$ for n = 2 and 3 we get

$$25+27-30\sqrt{3}>0, \text{i.e.}\ 26-15\sqrt{3}>0 \qquad \ldots(35)$$

and,

or $$\sqrt{3}>265/153 \qquad \ldots(36)$$

Again from (35) we have, by squaring

$$\left(26-15\sqrt{3}\right)^2>0$$

or, 676 + 675 – 780√3 >0, i.e. $\sqrt{3}<1351/780$ $\sqrt{2}=1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}$ $\left(\ldots 3\sqrt{3}\right)<0$

which along with (36) leads to the remarkable result (34). Thus by just squaring twice and cubing once, the baffling Archimedean limits in (34) are obtained!

But we still need an explanation for the initial choice of the fraction 5/3 instead of, say, 3/2 or 7/4. My attention goes to the ancient rule:

…(37)

with a = 1 and x = 2. This rule was also used in India.[30] It gives a lower value to allow a convenient positive correction to be made.

Datta in his famous classical work *The Science of the Sulba*[31] gives a plausible geometrical derivation of the following 4-term expressions

…(38)

$$\sqrt{3}=1+\frac{2}{3}+\frac{1}{3.5}-\frac{1}{3.5.52} \qquad \ldots(39)$$

We write these as

$$\sqrt{2} = \left(\frac{17}{12}\right) - \frac{1}{408} = \frac{577}{408} \qquad \ldots(40)$$

$$\sqrt{3} = \left(\frac{26}{15}\right) - \frac{1}{780} = \frac{1351}{780} \qquad \ldots(41)$$

It is seen that the first two terms of (39) gives the starting fraction 5/3 used by Thiele in deriving the Archimedean result (34) whose upper limit is in fact the above value of √3 as shown by (41). Also 5/3 comes from (37).

The approximation (38) is found in all the three major *śulba-sūtras*.[32] as is well known. Its first two terms can also be attained by using (37) and represent the fraction 4/3. It can be easily seen that (4-3√2)<0.

So by expanding $(4-3\sqrt{2})^n$, successively we get

$$\left(4-3\sqrt{2}\right)^2 = 16+18-24\sqrt{2} = 34-24\sqrt{2} > 0 \qquad \ldots(42)$$

$$140/99 < \sqrt{2} < 577/408$$

$$\left(4-3\sqrt{2}\right)^3 = 280-198\sqrt{2} < 0$$

$$\left(4-3\sqrt{2}\right)^4 = 4\left(17-12\sqrt{2}\right)^2 \qquad \ldots(43)$$

, by (42);

$$= 4\left(289+288-408\sqrt{2}\right) > 0 \qquad \ldots(44)$$

From (43) and (44) we get

Here also the upper limit represents the *śulba* value.

Interesting by (35) and (42) also lead as to the popular and convenient approximations √2 = 17/12 and √3 = 26/15. It may be noted that (40) and (41) show that a small negative correction in each of these convenient fractions lead us to the good and final values as implied in (38) and (39). More significant to note are their associated relations

$$2.12^2+1 = 17^2, \text{ and } 3.15^2+1 = 26^2. \qquad \ldots(46)$$

These relations at once show their connection to the well-known Indian *varga-prakṛti* equation

$$Nx^2 + 1 = y^2 \qquad \ldots(47)$$

Further by writing the first equation in (46) as

$$\sqrt{2} = (1/12).\sqrt{17^2 - 1}$$

And applying the formula (30) with x = -1, we get

which is equivalent to (40). Other ancient methods such as Heronian algorithm and iteration also lead us to same solution equivalently.[33]

An important point to note is that just as the pair (12, 17) is a solution of (47), the pair (408, 577) picked up from (40) is also a solution because

$$2.408^2 + 1 = 577^2 \qquad \ldots(48)$$

This also means tat the solution (408, 577) could have been derived from (12, 17) by using the *tulya-bhāvanā* theorem of Brahmagupta (628 AD). What we have shown is that the task is done just by squaring i.e. by $(17\text{-}12\sqrt{2})^2 = 0$.

$$\sqrt{2} = (17/12) - 1/12.2.17$$

5. Method of Averaging

In the history of mathematics, the use of average (arithmetic mean of two or more numbers or measures) has frequently yielded helpful results. This was especially so in those cases where the exact results were not known or were cumbersome to derive. Of course, in many cases the exact mathematical formula itself is nicely expressed in terms of certain average. The sum of an arithmetical progression is the average of the first and the last term multiplied by the number of terms. The area of a trapezium is equal to the product the average of the parallel sides and their distance.

In practice, a compelling situation far employing the technique of averaging arose in antique time when the problem of finding the area of general quadrangular field was faced. The quadrilateral of sides a, b, c, d is physically fixed on the ground but mathematically the four sides are not enough to fix the figure or define its area uniquely. So it is not possible to find an exact mathematical formula for the area in terms of four sides alone. Moreover enough sophisticated mathematics could not be expected to the known in remote antiquity.

We might see averaging of the four sides in the primitive Jorge's rule (11) for the area of a quadrilateral. Mathematically more analytic ancient peoples solved (3000 BC or earlier) the problem by using the formula

Area, $A = [(a + c)/2]\ [(b + d)/2]$ …(49)

which simply takes the product of the average length and average breadth. The formula (49) was so popular that it is found widely used in almost all ancient civilizations.[34] It is variously called as Surveyor's Rule or Taxman's formula or Adāo's Method, and is said to be used even now in the absence of a convenient practical rule. It always overestimates the area of all quadrilateral except rectangles. It helped in maintaining a sort of uniformity of practice and calculation.

In India, the first explicit statement of (49) is found in the *Brāhmasphuṭa-Siddhānta* XII. 21 of Brahmagupta as a rough rule.[35] An interesting point to note is that (49) was often used for triangles also by assuming one side of quadrilateral to be zero. But this will lead us to three results (in the case of a general triangle of sides a,b,c) namely

$(a+c)\ b/4$, $(b+c)a/4$, and $(a+b)c/4$.

However, by using averaging technique here, we can get the unambiguous rule for the triangle as

Area = $(ab+bc+ca)/6$ …(50)

For the case d=0, i.e. for a triangle Jorge's formula (11) will give

Area = $(a+b+c)^2/16$ but a better suggested rule is:

Area = $(a+b+c)^2/21$ …(51)

For an equilateral triangle (51) gives area 0.429 a^2 (correct answer 0.433 a^2 nearly), but for the triangle of sides13,14, and 15, the formula (51) yield exact area of 84 units.[36]

It seems that averaging was taken to be more convenient even when better results could be found by other methods. For the area of a drum-shaped field (double trapezia), the 4th country AD. Chinese compilation *Wu Tshāo Suan Ching* gives the formula[37]

Area = $[(a+b+c)/3]$. (height) …(52)

where a and c are the two parallel edges of the field and b is the linear measure along the line lying half-way between the above edges. The correct formula

Area = [(a+c)/2 +b}.h/2 …(53)

was used in India by Bhāskara I (629 AD).[38]

A point to note in this connection is that the process of averaging may lead to exact results accidentally when only an empirical one is expected. For instance, suppose a cone, a hemisphere, and a cylinder (all of same height r) are described on the same circular base (of radius r). Assuming the hemisphere to be the average between cone and cylinder its volume will be[39]

$$\text{Vol} = (1/2)\ [(\pi r^2.r)/3 + \pi r^2.r.]$$

$$= (2/3)\ \pi r^3$$

which is, in fact, the exact volume of the hemisphere. If the base and top of a frustum-like solid are rectangles (of sides a, b and a′, b′) with edges of top also parallel to edges of base correspondingly, them we have two averaging type formulas for volume of the solid

$$V_1 = (1/2)\ (ab + a'b').h \qquad \ldots(54)$$

$$V_2 = [\ (a+a')/2].\ [(b+b')/2].h \qquad \ldots(55)$$

where h is the height of the solid.

For frustum (*dvādaśāśra*) of square base and top, both the above formulas were used in Babylonia.[40] Brahmagupta calls V_1 which is based on averaging the areas, as *autra* or *auṇḍra* (gross) volume and V_2 which is based on averaging first the linear dimension (of base and top) as *vyavahārika* (practical) volume. He then uses the concept of a sort of weighted average to reach the final volume of the frustum.[41] An elegant generalization by considering any number of sections (instead of mere base and top) was given two centuries later by Mahāvīra[42] in his *Gaṇita sāra-saṅgraha* VIII, 9-11.

For a truncated right triangular prism (*navāśra*) the following averaging was used in China[43]

$$\text{Volume} = (\text{base})\cdot(h_1+h_2+h_3)/3^*$$

* This ancient Chinese formula was found to be mathematically exact by A.M. Legendre in his *Eléments de Géométrie* (1794).

where h_1, h_2, h_3 are the heights of the vertical edges, and similar process applied to other irregular solids.

Mention of Heronian algorithm for finding square root has been made already. This popular ancient method is based on averaging. For finding $\sqrt{N}$, we take any (rough) approximation a_0. Then N/a_0 is another approximation. The average (arithmetic mean) of the two is the next (better value). For example, assuming 17/12 to be the starting near about value for $\sqrt{2}$. Then 2/(17/12) or 24/17 is another value. And

$$(1/2\ (17/12 + 24/17) = 577/408$$

will be a better approximation than 17/12. In fact it is the *Śulba* approximation in the form (40). For better value of $\sqrt{2}$, we can repeat the process with a_0 = 577/408. Heronian method always yields value in excess of the true value. So in vain Neugebauer and Sachs tried to explain by it the Babylonian $\sqrt{2}$ = 1; 24, 51, 10 which is in defect.[44] Recently D.G. Morin of Venezuela has extended Heron's algorithm to cube roots etc. by averaging of rational means.

Consider now the approximation $\sqrt{a^2 + x} = a + x/(2a - c)$ $\sqrt{a^2 + x} = a + 2x/(4a + 1)$

$$\sqrt{N} = \sqrt{a^2 + x} = a + x/(2a + c) \qquad \ldots(49)$$

which gives result in excess (30) when c = o, or in defect (37) when c = 1. For the average or mean value of c (i.e.c. = ½) we get[45]

$$\ldots(50)$$

which is found in al-Uqlīdisī (10th century).

A similar discussion can be given for the rule

$$\ldots(51)$$

In Jaina cosmography, the circular *Jambādvīpa* is divided into a number of segments by parallel chords. For the area of a segment between two chords (lying on the same side of the centre) of lengths a and b, Jina Bhadra Gaṇi (c. 600 AD) gave the formula

$$A = [(a+b)/2]\cdot(\text{height}) \qquad \ldots(52)$$

which is based on the average of a and b. He knew the gross defectiveness of (52), and so gave another rule which used root-mean square of a and b.[46]

In the absence of exact methods, the “best” that Kepler (c. 1600) could do for the perimeter of an ellipse was to take average of the circles on the two axes[47]. For its area a medieval formula took the average of the axes and computed the area by using[48]

$$\text{Area} = \pi[(a + b)/2]^2$$

This is surprising because the correct formula π ab could have been reached in a simple manner. Averaging has been also used in finding $\pi = 355/113$ by Adriaen Anthoniszoon in 1585 and in connection with certain series[50]. The use of arithmetic mean for averaging is justified by the Principle of Least Squares.

6. The Golden Rule of Three (Trairāśika)

In some form or the other, the *trairāśika* (Rule of Three) is being used universally since remote antiquity. It was called a Golden Rule due to its simplicity and utility in all practical matters of calculation. The importance of the rule is mentioned by the famour Bhāskarācārya by saying that “as the lord Hari pervades the universe with His manifestation so does The Rule of Three, with is variations, pervades the whole science of calculation”.[51]

The Rule of Three is a basic rule of Arithmetic. It is frequently used in other branches of mathematics either as such or in it other forms. Also, Rule of Five, Rule of Seven etc. are used as its higher forms when the number of variables is more.

When similarity property is used in geometry, it amounts to using the Rule of Three. Similar triangles ABC and PQR gives (Fig. 3).

$$AB/PQ = AC/PR \qquad \dots(53)$$

If any three segments in (53) are known we can find the remaining 4th segment e.g.

$$PR = (PQ/AB).AC \qquad \dots(54)$$

In the language of Rule of Three, we may say: “Given AB, we get PQ. Then how much or what shall we get when AC is given.” The answer is PR represented by (54).

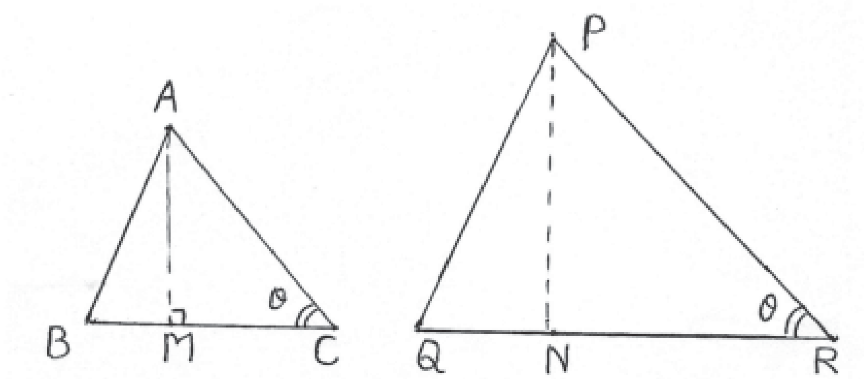

Fig. 3.

In trigonometry, the similarity or proportionality property is indicated by certain named functions. For instance the ratio AM/AC is called sine of angle ACM and is written as sin θ which will also be equal to the ratio PN/PR in the similar triangle PRN. Thus when we use trigonometrical function, it implies the use similarity property geometrically and the Rule of Three arithmetically as shown alone.

In the case of simple or linear interpolation also, we are essentially using the Rule of Three by taking proportionate changes in the argument and the functional values. For example, let us find sine of 35° from known sines of 30° and 45°. Here we have (upto 4 decimals).

$\sin 30° = 0.5000$

$\sin 45° = \sqrt{2}/2 = 0.7071$

Change: $+ 15° = +0.2071$

Here, change of 15° in angle corresponds to a change of 0.2071 in the value of sine (approximately).So by Rule of Three, for 5° (from 30° to 35°), the change in sine value will be (linear interpolation)

$= (5/15) \times 0.2071 = 0.0690$ nearly

Hence, $\sin 35° = \sin 30° + 0.0690 = 0.5690$.

Thus we find that the Rule of Three in the more general and wider sense is used in various forms. It has been used as a method of proof as well as of computation through out the history of mathematics.

The popular algebraic formula (37) is based on the Rule of Three (see section 4 above). Suppose the non-square positive integer N is $(a^2 + x)$,

where $0 < x < (2a + 1)$. Let

Now we see that when c = 0, e is also 0. But when c = 2a+1, e will be 1. so when c = x, e will be x/(2a+1) by the Rule of Three applied empirically. Hence we have

…(55)

By using this we get

…(56)

It is interesting to mention that al-Bīrunī credits Brahmagupta for deriving (56) and for knowing 22/7 as an approximation of π for which the latter used √10 as the accurate value[52] for n^{th} root, the corresponding empirical formula will be

$\sqrt{N} = \sqrt{a^2 + c} = a + e$ $\sqrt{a^2 + x} = a + x/(2a+1)$ $\sqrt{10} = \sqrt{3^2 + 1} = 3 + 1/7 = 22/7$

$(a^n + x)^{1/n} = a + x / [(a+1)^n - a^n]$ …(57)

The case n = 3 was used by Leonardo Fibonacci perhaps for the frest time (c. 1220)[53]. In fact in this case, the denominator of the second term in (57) can be variously taken as $(3a^2 + 3a + 1)$, or $(3a^2 + 3a)$, or $(3a^2 + 1)$, or $3a^2$ in decreasing order. And it is noteworthy that the rules for cube root with all four expressions in (57) are found in various Arabic and European authors[54].

In India Lakṣmīdāsa Miśra (c. 1500) made a very peculiar use of the Rule of Three. Bhāskara II in his *Jyotpatti* (9^{th} verse) had given the exact value

$$R \sin 18° = \left(\sqrt{5R^2} - R\right) / 4 \qquad \text{…(58)}$$

To prove this, Miśra started with R sin90° = R and wrote it as

$$R \sin 90° = \left(\sqrt{25R^2} - R\right) / 4 \qquad \text{…(59)}$$

He argued that for the sine of 90°, the coefficient of R^2 in (59) is 25.

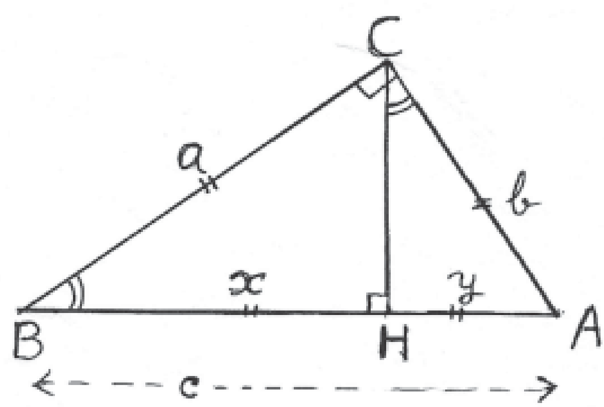

Fig. 4

Hence for sine of 18°, this coefficient, by Rule of Three, should be (18/90) x25 i.e 5 and thus we reach the result (58)! Of course, his argument is very empirical as it will not work for other angles (say, 30°) as was pointed by Munīśvara in his *Marīci* on *Jyotpatti*.[55]

For the so-called Pythagoras Theorem, a very short proof was given by Bhāskara II (12th century) by using the similarity property or the *trairāśika* (Rule of Three) as he calls it[56]. In his *Bījagaṇita* ("Algebra"), he considers the similar triangles ABC (the given right angled triangle), CBH, and ACH, CH being perpendicular from C on the hypotenuse AB. We have

$x/a = a/c$, and $y/b = b/c$

By putting x and y from these in $x+y = c$, and simplifying, we at once get the required result

$$a^2 + b^2 = c^2 \quad \ldots(60)$$

The similarity of the above three triangles (Fig. 4) can lead us to another proof if we use the fact that areas of similar figures similarly described on their bases, are proportional to the squares of the bases. Now the ΔABC is described on the hypotenuse AB = c, ΔBCH on BC = a and ΔACH on AC = b, and since the area of the biggest triangle ABC is equal to the sum of the areas of the other triangles, the result (60) follows. This proof was given by H.A. Naber in 1908[57].

About Bhāskara's above short proof, Cajori[58] says that it "was unknown in Europe until it was rediscovered by Wallis who gave it in his treatise on angular sections[59]. But according to Loomi[60], it also appears in Fibonacci's *Practica Geometriae* (1220).

In India, The technique of the Rule of Three has been suggested in deriving the formulas which may not be considered simply elementary. One such result is Bhāskara I's remarkable formula

$$\sin\theta = 4\theta\,(180-\theta)/[40500-\theta(180-\theta)] \quad \ldots(61)$$

found in his *Mahābhāskarīya*, VII, 17-19 (7th century)[61]. In (61) the angle is in degrees and it represents a rational approximation to a transcendental function. An equivalent geometrical form of (61) appears in the *Līlāvatī* (rule 210) of Bhāskara II whose famous commentator Gaṇeśa (1545 AD) remarks[62]

yathakathaṃicita trairāśikam-upalabdhyā ācāryaiḥ kalpitam

"Some Rule of Three was applied by the professors to obtain the result".

Now formula (61) can be written in the simpler form as

$$\sin\theta = 4P/(5-P) \quad \ldots(62)$$

where

$$P = \theta\,(180-\theta)/8100 \quad \ldots(63)$$

We note that (62) is better than the parabolic approximation

$$\sin\theta = P \quad \ldots(64)$$

The behaviour of sin θ, P, and P.sinθ is similar e.g. all vanish for θ = 0 and 180, and all attain the same greatest value at θ = 90 about which they have a symmetry. For θ =30, there value is 1/2, 5/9, and 5/18 respectively. So by Rule of Three (i.e. linear proportionality) applied to deviations of P and P. sinθ from sinθ, we have[63]

$$\frac{(P-\sin\theta)}{(P.\sin\theta-\sin\theta)} = \frac{\left(\frac{5}{9}-\frac{1}{2}\right)}{\left(\frac{5}{18}-\frac{1}{2}\right)}$$

which on simplification, yields (62) .

More remarkable is the Indian proof of the second order property of the sine function viz. that the second order finite differences of sines are proportional to the sines themselves. The proof is intelligently creative and uses simply the Rule of Three twice[64]. The golden rule has been also used in solving problems of spherical astronomy by the technique of 'working inside the sphere[65].

7. Representation and Approximation by Unit Fraction

Unit fractions were quite popular in ancient times among various civilizations along with the sexagesimal fractions. In the *Maitrāyaṇī Saṃhitā* 3.7.7 of the *Kṛṣṇa Yajurveda*, the fractions 1/4, 1/8, 1/12 and 1/16 are called *pāda*, *śapha*, *kuṣṭha and kalā* respectively and some of these names had already appeared in the *Ṛgveda*[66].

It was in Egypt that the unit fractions were used extensively as is clear from the famous Rhind Papyrus (c. 1650 BC) which was copied by Ahmes (or Ahmos) from an still older document. In fact Egyptian scholars took great pains in preparing tables of unit fractions and in expressing various results in terms of unit fractions. For example, consider problem 31 from the above papyrus : 'A quantity, its 2/3, its 1/2, its 1/7 together make 33. what is the quantity?' In modern form[67] the problem is to solve (x being the unknown quantity)

$$(2x/3) + (x/2) + (x/7) = 33 \qquad \ldots(65)$$

The answer (i.e. value of x) is given in the complicated form as

$$14+\frac{1}{4}+\frac{1}{56}+\frac{1}{97}+\frac{1}{194}+\frac{1}{388}+\frac{1}{679}+\frac{1}{776} \qquad \ldots(66)$$

It is clear that cumbersome labour was done for love of the unit fractions, the modern solution has the form 14 28/97, and even this can be expressed as

$$14+\frac{1}{4}+\frac{1}{26}+\frac{1}{5044} \qquad \ldots(67)$$

But a merit of (66) is that numbers used are all below 1000.

With the same merit, the Rhind Papyrus contains a table in which fraction of the form 2/N are expressed in terms of unit fractions for all odd values from N = 5 to N = 101 e.g.

$$\frac{2}{17}=\frac{1}{12}+\frac{1}{51}+\frac{1}{68} \qquad \ldots(68)$$

$$\frac{2}{101}=\frac{1}{101}+\frac{1}{202}+\frac{1}{303}+\frac{1}{606} \qquad \ldots(69)$$

The table is remarkable, beautiful and shows mathematical feat. There is no arithmetical error, and we cannot fail to appreciate that each expansion sets the fractions in descending order of magnitude without repetition.

In this connection, it must be remembered that the representation of a fraction in terms of unit fractions in not unique in the light of relations like

$$\frac{1}{n} = \frac{1}{(n+1)} + \frac{1}{n(n+1)} \qquad \ldots(70)$$

For instance we have

$$\frac{2}{5} = \frac{1}{3} + \frac{1}{15} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20} = \frac{1}{5} + \frac{1}{6} + \frac{1}{30}$$, etc.

But it is interesting to note that if we confine to expansions of 4 terms and use numbers upto 1000, then (69) is unique.

A simple and practical algorithm to express a given fraction p/q into unit fractions is the Mahāvīra – Fibonacci method[68]. In this method the denominator q is slowly increased by 1, 2, 3, … till we reach a value x such that (q + x) just becomes a multiple (say r times) of p, so that

p/q = p/(q+x) + [(p/q) – p/(q+x)}

= 1/r + (p x/r)

We repeat the process with the second term (px/r) if it is not already a unit fraction etc.

Exm. 1: $\frac{2}{17} = \frac{2}{18} + \left(\frac{2}{17} - \frac{2}{18}\right) = \frac{1}{9} + \frac{1}{153}$ which is different from (68).

Exm. 2: $\frac{7}{9} = \frac{7}{14} + \left(\frac{7}{9} - \frac{7}{14}\right) = \frac{1}{2} + \frac{5}{18}$

$$= \frac{1}{2} + \frac{5}{20} + \left(\frac{5}{18} - \frac{5}{20}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{36} \qquad \ldots(71)$$

Exm. 3: $\frac{11}{17} = \frac{11}{22} + \left(\frac{11}{17} - \frac{11}{22}\right) = \frac{1}{2} + \frac{5}{34}$

$$= \frac{1}{2} + \frac{5}{35} + \left(\frac{5}{34} - \frac{5}{35}\right) = \frac{1}{2} + \frac{1}{7} + \frac{1}{238} \qquad \ldots(72)$$

There is another general method which has been called Vedic Principle. It is based on minimality property and admits both positive and negative terms. It can be applied to expand fractions (p/q) as well as to other numerical quantity Q (e.g. surd √N) in terms of converging unit fractions. In this method of trial of successive terms the assumed form (finite or infinite) has the pattern

$$Q = I \pm \frac{1}{n_1} \pm \frac{1}{n_1.n_2} \pm \frac{1}{n_1.n_2.n_3} \pm ...... \qquad \ldots(73)$$

We first find integer I nearest to Q. Then we add or subtract from it a until fraction ($1/n_1$) such that ($I + 1/n_1$) is nearest to Q. Then again add or subtract from this resulting rational number, a unit fraction ($1/n_2$) times or multiple of the last unit fraction ($1/n_1$) such that the new resulting rational number namely

$I \pm (1/n_1) \pm (1/n_1.n_2)$

is closest to Q. And so on by repeating the process if necessary. Here the obtained expansion will represent the best or closest value at any stage.

Example 1 : Express 2/101 in terms of expansion of the type (73). Here 2/101 = 1/50.5, and thus 2/101 lies between 1/50 and 1/51. Now considering the deviations, we see that

$(1/50) - (2/101) = +1/(50 \times 101)$

and (1/51) – (2/101) = -1/(51 x 101) which is numerically less than the above deviation. Thus the unit fraction 1/51 is nearest to 2/101 e.g. $n_1 = 51$ in (73) and with I = 0, we now write

$2/101 = (1/51) + 1/(51 \times n_2)$

Luckily in this example, the above second deviation also readily tells us that $n_2 = 101$ will give the exact value

$2/101 = (1/51) + 1/(51 \times 101)$ …(74)

It can be easily seen that (74) can be also obtained by the Mahāvīra-Fibonacci method but the Egyptian (69) is quite different.

Example 2: To represent 7/9 in form of (73).

Here 7/9 = 1/(9/7) = 1/(1.3), nearly. So 7/9 lies between 1/1 and 1/2, and considering the deviations, we have

1 - (7/9) = 2/9, while (1/2 – (7/9) = - 5/18

which is numerically greater than the first. So we take I = 1 and write

$7/9 = 1 - (1/n_1)$ …(75)

From this, $1/n_1 = 2/9 = 1/4.5$, so that n_1 is 4 or 5.

With $n_1 = 4$, the deviation from 7/9 will be

= 1 – (1/4) – (7/9) = –1/36

With $n_1 = 5$, The deviation will be, by (75)

= 1 – (1/5) – (7/9) = +1/45 …(76)

which is smaller, so that $n_1 = 5$ is to be accepted in (75) and we have

7/9 = 1 – (1/5)

Also if $(-1)/(5 \times n_2)$ is the next term, (76) shows that $n_2 = 9$. Thus

7/9 = 1 – (1/5) – 1/(5•9) exactly …(77)

It should be noted that 'Vedic' expansion (77) is different from (71) obtained by the simpler Mahāvīra-Fibonacci algorithm. The Vedic method is based on minimality principle and gives 'best' term by term expansion.

Example 3 : Expand 11/17 into unit fractions by Vedic method. Here 11/17 = 1/(1.55), nearly and it can be seen that it lies nearer ½ than 1. So we assume now

11/17 = (1/2) + (1/2n) …(78)

In which n stands for n_2, while $n_1 = 2$. From this we get

1/n = 2[(11/17) – (1/2)] = 5/17 = 1/3.4

So we have to check (78) for closeness for n = 3 and 4 (between which n lies). For n = 3, the deviation of the right hand side of (78) from 11/17 is seen to be 1/51 while that for n = 4 is found to be (-3)/136 which is numerically greater. Also, the first deviation is a unit fraction and we have

$$11/17 = (1/2) + (1/6) - (1/51) \qquad \dots(79)$$

However, it must be noted that it is not of the type (73) because 51 is not a multiple of 6. So we write

$$11/17 = (1/2) + (1/6) \pm (1/6m) \qquad \dots(80)$$

in which $n_1 = 2$, $n_1 . n_2 = 6$, and m stands for n_3

Now from (80)

$$\pm 1/m = 6[(11/17) - (1/2) - (1/6)] = (-2/17) = -1/8.5$$

So we have to take lower sign in (80) and test for m = 8 and 9. It can be seen that m = 9 gives the right hand side of (80) closer to 11/17, than m = 8. In fact, taking lower sign and m = 9, the numerical error in right hand side of (80) is found to be 1/17x54.

Hence we exactly have

$$11/17 = (1/2) + (1/6) - (1/54) - (1/918) \qquad \dots(81)$$

which is the required Vedic expansion of the type (73) but is quite different from (72) in which only positive unit fractions are used. Both have their own merit and so also (79) which has smaller numbers.

Afzal Ahmad[69] has used the above minimality method implied in expansion of the type (73) in connection with approximating simple surds. He has successfully shown that the well known *śulba* value (38) of √2 follows by applying this Vedic principle. By this principle, he extended the *śulba* value to

$$\sqrt{2} = 1 + (1/3) + (1/3.4) - 1/(3.4.34) - 1/(3.4.34.1154) - 1/(3.4.34.1154.1331714) \qquad \dots(82)$$

Moreover, he has supplied a theoretical proof of the Vedic principle and added many details.[70]

When the above principle is used to approximate √3, we get[71]

$$\sqrt{3} = 2 - (1/4) - 1/(4.14) - 1/(4.14.194) \qquad \dots(83)$$

This is quite different from (39) which Datta derived by using the method he used for √2. In this connection an important point may be mentioned. The 'Vedic Principle' approximation (83) yields, by taking 1,2,3 and all 4 terms, the values

$\sqrt{3}$ = 2/1, 7/4, 97/56, and 18817/10864 respectively. Each of these fractions may be denoted by y/x by picking up numerator of denominator, it will be found that in each case (x,y) is a solution of the famous *varga-prakṛti* equation

$$Nx^2 + 1 = y^2 \text{ for } N = 3 \qquad \ldots(84)$$

In case of (82) (i.e. N = 2), this equation is not satisfied by first two approximations (1/1 and 4/3) but is satisfied by other approximations 17/12, 577/408, 665857/470832, etc. As the pair (2,3) is a solution of (84) with N = 2, the first three terms in (82) can also be replaced by

$$1 + (1/2) - 1/(2.6), \text{ or } 3/2 - 1/(2.6)$$

However, this will not be strictly according to the conditions of the principle of (73). The choice of 3/2 (in place of 4/3) violates minimality.

The problem of expanding any arbitrary positive number N in the form (73) was considered by J.H. Lambert[72] in 1770 such that the series should converge as rapidly as possible. Expansions in terms of continued fractions were also developed in ancient and medieval times. In addition to unit fractions, the sexagesimal (or astronomical) fractions were used in expansions.

8. Miscellaneous

The Earth is spherical, yet due to its large radius, a small region on it look plane. Similarly in a relatively big circle, small arcs of it will look as straight lines. The traditional *Brahma Siddhānt*a (Śākalya) I. 93 says[73]

vṛttasya ṣaṇṇavatyaṃśo daṇḍavat

"The 96th part of a circle is (straight) like a rod."

That is, in a circle of radius R and circumference C, the small arc s measuring C/96 or 3°45′ (=h) in angular units, is taken equal in length to the straight chord AB approximately (Fig. 5). Since FA is small here, the sine-chord BF or R sin h is also taken equal to arc s (= h in angular units) empirically. In ancient Indian trigonometry many rules and tables are based on this initial assumption.

The *Āryabhaṭīya* is supposed to be the historically first work of the dated-type (*pauraṣeya*) which has a sine table for *Sinus totus* R = 3438′ and tabular interval h = 90°/24 = 225 minutes. Actually, instead of the 24 tabular Sines

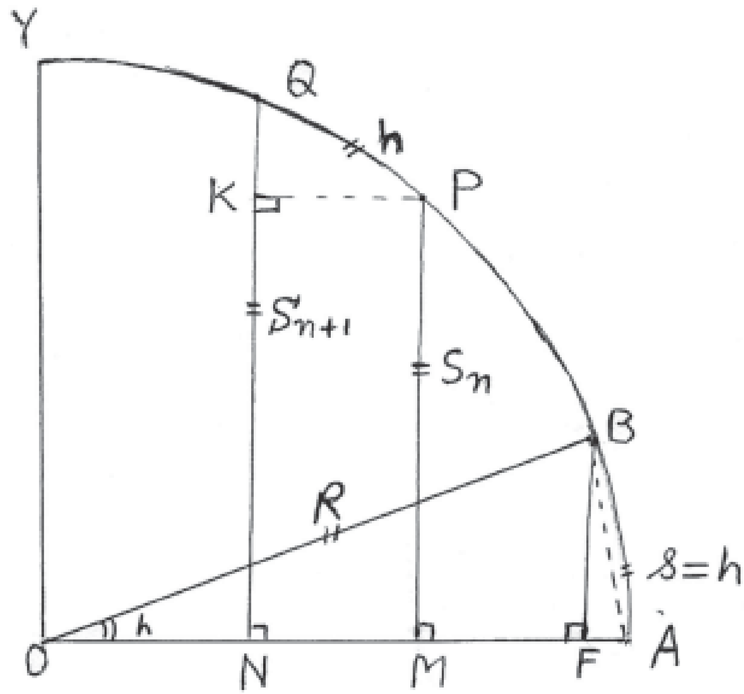

Fig. 5

$S_n = R \sin nh,\ n = 1,2,3,$ …(85)

Āryabhaṭa gives their 24 Sine Differences

$D_n = S_n - S_{n-1}$ (S_o being zero) …(86)

From this we have

$S_{n+1} = S_n + D_{n+1}$ …(87)

The numerical values of Dn are (*Āryabhaṭīya*, I. 10)

225, 224, 222, 219, 215,…..37, 22, 7 …(88)

A cryptic-type short rule using which the set (88) seems to have been obtained is found in the work (II.12) and may be presented as follows:

$D_1 = S_1 = 225$

$D_2 = D_1 - (D_1/ D_1) = 224$

$D_3 = D_2 - (D_1/ D_1 + D_2/ D_1) = 222$ etc.

In general

$$D_{n+1} = D_n - \sum_1^n (D_n) / D_1 \quad \ldots(89)$$

$$= D_n - S_n/S_1 \quad \ldots(90)$$

Using (89), Ayyangar[74] has already worked out the set (88) and has explained many discrepancies.

The exact mathematical form of (89) as given by Nīlakaṇṭha Somayaji (c. 1500) is [75]

$$D_{n+1} = D_n - S_n.(D_1- D_2)/ D_1 \quad \ldots(91)$$

For the factor $(D_1- D_2)/ D_1$, Āryabhaṭa took 1/225, but its exact mathematical expression (independent of R)) is

$$2 (1\text{-}\cos h) = 1/233.53, \text{ for } h = 225' \quad \ldots(92)$$

Bhaskara II in his *Jyotpatti* (verses 19-20) has given the following values[76]

S_1 = R sin h = 225 – (1/7)

and (R cos h)/R = 1 – (1/467)

Using cos h from these, the denominator in (92) will be found to be 233.5 which is quite near the closer accurate value mentioned there, Nīlakaṇṭha also gives the same value 233.5 while his commentator Śaṅkara Vāriar gives the still better value as[77]

233 + 32/60

A technique to improve empirically obtained certain rough results was that of *asakṛta-karma* (repetitive process) or iteration. An ancient Indian case may be cited in this context. For finding the value of Sine for any intermediary argumental value $x = ph + \theta$, which lies between p^{th} and $(p+ 1)^{th}$ tabular values, the linear interpolation rule gives

$$R \sin x = R \sin ph + [R \sin (ph+h) - R \sin ph].(\theta/h) \quad \ldots(93)$$

$$= S_p + (\theta/h).\ D_{p+1} \quad \ldots(94)$$

where D_{p+1} is the current (*bhogya*) tabular difference.

For better result, Brahmagupta has given an expression, D_t, based on second order finite differences, to be used in place od D_{p+1} in (94) as follows (in present notation)

$$D_t = (1/2)\ (D_p + D_{p+1}) - (\theta/2h)\cdot(D_p - D_{p+1}) \qquad \ldots(95)$$

This is called *bhogyam* or true tabulr difference for any. Now for the inverse interpolation, that is, for finding θ when R sin x is given, we have, from (94) after replacing D_{p+1} by D_t,

$$\theta = [R \sin x - R \sin ph].h/D_t \qquad 96$$

But, since itself is unknown, we cannot find Dt from (95). So, Brahmagupta prescribes what is called the *asakṛta-karma* or iteration in his *Khaṇḍa Khādyaka*[78]. By taking D_{p+1} in place of D_t in (96) we get the first approximation θ_1 (of θ). Putting this θ_1 in (95) we get an initial value of D_t which we use in (96) to find a better value θ_2. By using θ_2 in (95) we get a better value of D_t which will yield still better value of θ (say θ_3) from (96). And so on.

Example: Find the angle whose R-sine is 61 from Brahmagupta's following small table (R = 150, h = 15°)

Angle :	15°	30°	45°	60°	75°	90°
Sine :	39	75	106	130	145	150= R
Sine-difference :	39=D_1	36	31	24	15	5=D_6

Here the given Sine value 61 lies between S_1 = 39 and S_2 = 75. So the angle lies between 15° and 30° and p = 1

By linear interpolation rule (94) or by (96) with D_t replaced by $D_{p+1} = D_2 = 36$ here, we get initial value

$$\theta_1 = (61- 39)\ 15/36 = 55/6$$

Using this for θ in (95) we have

$$D_t = (1/2)\ (39+36) - [55/(6 \times 30)]\ (39-36) = 439/12$$

By putting this in (96), the better value of θ is obtained as

$$\theta = \theta_2 = (61-39).\ (15 \times 12/439) = 9.02 \text{ nearly.}$$

With this value of θ, the required angle is

$$X = ph + \theta_2 = 15 + 9.02 = 24.02° = 24°\ 1'.2.$$

If we do one more iteration i.e. put θ_2 in (95) and then put the resulting better D_t in (96), we will get the still better value of θ, namely 9.017 degrees. This leads us to the accurate answer x = 24 0′ 6″ nearly which is almost equal to correct value that is almost equal to 23°59′45″.

There is an important point in the context here. The direct method of finding will be to put D_t from (95) into (96) and solve the resulting quadratic equation in θ. This algebraic method is called *bījakarma*.[79] In the case of above example, the quadradic equation will be

$$\theta = \frac{3300}{(375 - \theta)}$$

whose relevant one root will be θ = 9.017 nearly which is same as the value of θ_3 found above.

Later on the iteration method was used in computing sine of A/3 and A/5 from given sin A.[80]

A very peculiar empirical rule for finding the sine of any angle quickly is found in Muñjala's *Laghu Mānasa* (932 AD), II.2, as follows[81]

catustryekaghna rāśyaikyaṃ bahukoṭyoḥ kalāṃśakāḥ

"The sum of factors 4, 3, 1 for the (respective three) signs represents the degrees and minutes of the sine and cosine".

That is, 4°4′, (4+3)° (4+3)′, and (4+3+1)° (4+3+1)′ are the sines of 30°, 60° and 90° respectively, the last one 8°8′ being the *sinus totus* or radius.

Example (i): Sin 24° = (4×24/30)° (4×24/30)′ = 3°5′12″, by taking proportional parts in the first sign in which 24° lies.

Example (ii): Find Sine of 75°36′ by Muñjāla's rule. Here 75°36′ lies in the 3rd sign, covering first two signs fully. So the sum of factors will be

= 4 + 3 + 1 × (15°36′)/30° = 7.52

Hence the required Sine = (7.52)° (7.52)′ = 7°38′43.2″

the correct value being (8°8′). sin (75°36′) = 7°52′40″.

Rationale of the Rule: The sines of 30, 60, 90 degrees are as

(1/2): (√3/2):1, i.e. as (1/2) : (7/8) : 1

by taking the approximation √3 = 7/4 for which see the equation (83) in the last section. Thus the three sines are in the proportion 4:7:8 and their differences as 4:3:1 as taken by Muñjāla. The rule is rough but so simple.[82]

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