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HISTORICAL NOTES

SQUARE ROOTS IN THE *ŚULBASŪTRAS*

In his study of the *Śulbasūtras*, the approximations to $\sqrt{2}$ found in these texts led Datta to strongly suspect that ".....the rudiments of the theory of continued fractions were known to the early Hindus"[1]. An investigation into this question is the topic of the present article.

The approximation to $\sqrt{2}$ referred to by Datta[2] in this connection are:

(1) $$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} = \frac{577}{408}$$

(2) $$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2.6} = \frac{17}{12}$$

(3) $$\sqrt{2} = \frac{7}{5}$$

The first approximation is given in *Baudhāyana-śulbasūtra* 2.12, *Āpastamba-śulbasūtra* 1.6 and *Kātyāyana-śulbasūtra* 2.9; the second in *Mānava-śulbasūtra* 12.1; and the third in *Mānava-śulbasūtra* 2.7. All references to these texts refer to the edition of Sen and Bag [3].

Datta based the above statement on the fact that these approximations are all convergents (the eighth, the fourth and the third convergent respectively) in the continued fraction representation of $\sqrt{2}$:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}$$

How the approximations were found by the Vedic geometers is not stated in the *Śulbasūtras*, but we can safely ignore Kaye's argument that since the system of units in the *Baudhāyana-śulbasūtra* involves the ratios 1:3, 1:4, and 1:34 (according to the *Baudhāyana-śulbasūtra* 1.3, 1 *prādeśa* equals 12 *aṅgula* and 1 *aṅgula* equals 34 *tila*), the first approximation is ".... only an expression of direct measurement".[4] The accuracy of the approximation makes it certain that some method must have been employed to obtain it.

A number of scholars have presented suggestions as to how the first approximation was found, including Rodet[5], Datta[6], Ganguli[7], Gurjar[8], Katz[9], and Henderson[10]. Some of these methods are geometric, while others are algebraic. Rather than going through each method in detail, it suffices here to point out that underlying all of the methods, be they geometric or algebraic, is a procedure corresponding to carrying out one or more steps of the following algorithm.

Let A be a positive integer, the square root of which is to be computed. If a_0 is a first approximation to $\sqrt{A}$, then successive approximations, $a_1, a_2, a_3, \ldots\ldots$,

are found as $a_{n+1} = a_n - \dfrac{a_n^2 - A}{2.a_n}, n = 0,1,2.....$

If Newton's method is applied to the function $f(x) = x^2 - A$, the same algorithm is obtained.

None of the suggestions brought forward by the above mentioned scholars concerning which method the ancients used to find the first approximation go beyond what one can reasonably assume that the Vedic geometers were capable of. Therefore, leaving aside the question of whether a geometric or an algebraic approach was utilized, we will assume here that the method was essentially that of applying the above algorithm one or more times.

A few observations will be useful in the following.

Observation 1. If $a_0^2 \neq A$, then $a_n > \sqrt{A}$ for all $n \geq 1$. This result follows from

the fact that $a_1^2 = A + \dfrac{(a_0^2 - A)^2}{4.a_0^2} > A.$

Observation 2. Suppose that $a_0 = \frac{\alpha}{\beta}$, where α and β are two mutually prime positive integers which satisfy $\alpha^2 - A.\beta^2 = \pm 1$. The next approximation is then $a_1 = \frac{\alpha^2 + A.\beta^2}{2.\alpha.\beta}$, and the computation $(\alpha^2 + A.\beta^2)^2 - A.(2.\alpha.\beta)^2 = (\alpha^2 - A.\beta^2)^2 = (\pm 1)^2 = 1$ shows that the two positive integers $\alpha^2 + A.\beta^2$ and $2.\alpha.\beta$ solve the Pell equation $x^2 - A.y^2 = 1$. This is a special case of a result used by the later Indian mathematician Brahmagupta (b. 598 AD) to solve the equation $x^2 - A.y^2 = 1$ in some cases[11]. The important point here is that if the first approximation provides a solution to the equation $x^2 - A.y^2 = \pm 1$, then all subsequent approximations will provide solutions to the Pell equation.

Observation 3. Let $\frac{p_n}{q_n}$, where p_n and q_n are mutually prime positive integers, be the n^{th} convergent in the above continued fraction representation of $\sqrt{2}$. It is a well-known result from modern mathematics that the pairs of numbers (p_n, q_n), n = 1,2,3,....., are precisely the solutions to the equation $p^2 - 2.q^2 = 1$.

Together, the second and third observations give us the result that if a starting point $a_0 = \frac{\alpha}{\beta}$ provides a solution to the equation $x^2 - 2y^2 = 1$, then all subsequent approximations generated from it, i.e. $a_1, a_2, a_3, \ldots$ will be convergents in the continued fraction representations of $\sqrt{2}$.

If one desires to find anapproximation to $\sqrt{2}$ starting from a simple value, then two starting-points recommend themselves, namely $1\frac{1}{2} = \frac{3}{2}$ and $1\frac{1}{3} = \frac{4}{3}$. The Vedic geometers knew that $(1\frac{1}{2})^2 = 2\frac{1}{4}$ (*Āpastamba-śulbasūtra* 3.8) and there is no reason to doubt that they knew that $(1\frac{1}{3})^2 = 1\frac{7}{9}$ as well.

Since $3^2 - 2.2^2 = 1$, we know from the above that any approximation arrived at starting with $a_0 = 1\frac{1}{2}$ will provide a solution to the Pell equation and hence be a convergent in the expansion of $\sqrt{2}$ as a continued fraction. Using the algorithm with $a_0 = 1\frac{1}{2}$, we get $a_1 = 1\frac{1}{2} - \frac{(1\frac{1}{2})^2 - 2}{2.1\frac{1}{2}.} = 1 + \frac{1}{2} - \frac{1}{2.6}$, which is precisely the second approximation listed above.

Since $4^2 - 2.3^2 = -2$, the starting-point $a_0 = 1\frac{1}{3}$ does not provide a solution to the equation $x^2 - A.y^2 = \pm 1$, but since $a_0 = 1\frac{1}{3}$ gives

$a_1 = 1\frac{1}{3} - \frac{(1\frac{1}{3})^2 - 2}{2.1\frac{1}{3}} = 1 + \frac{1}{3} + \frac{1}{3.4} = \frac{17}{12}$ and $17^2 - 2.12^2 = 1$, we conclude that

every approximation obtained by using $a_0 = 1\frac{1}{3}$ as the starting-point will provide a solution to the Pell equation. Using the method twice, we get

$a_2 = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$, which is the first approximation.

In other words, given our assumption about the method by which the Vedic geometers arrived at the first approximation to $\sqrt{2}$ and the above starting-points, we see that it is not surprising that the first and second approximations are convergents in the continued fraction representation of $\sqrt{2}$. In fact, it could not be otherwise, and there is thus no need to assume that the Vedic geometers were familiar with the theory of continued fractions in order to explain this.

Concerning the third approximation, then, since $\frac{7}{5} < \sqrt{2}$, the first observation shows that it is not possible to arrive at the approximation $\frac{7}{5}$ using the above algorithm. However, the approximation $\frac{7}{5}$ is a simple one that may have suggested itself without the use of a method. It is not stated explicitly in the *Mānava-śulbasūtra*, but can be deduced from a rule for measuring out the *paitṛkī vedi* with a cord, and may therefore have functioned merely as a rule of thumb.

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