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COMPUTATION OF THE TRUE MOON BY MĀDHAVA OF SAṄGAMA-GRĀMA

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Saṅgama-grāma Mādhava (1400 AD) was a great astronomer and mathematician of his times. A number of his works have failed to survive but in the ones that survived and the mathematical contributions that have come to light through the quotations in later works, we find the signature of his ingenuity. The present paper is a study on his astronomical treatises, *Veṇvāroha* and *Sphuṭacandrāpti*, which contain the true longitudes of Moon derived by Mādhava, in the light of modern astronomical computations. Mādhava's simplest techniques for computing true moon in his *Veṇvāroha* or (ascending the Bamboo) based on the anomalistic revolutions of Moon is illustrated with examples and the results have been compared with the computer-derived modern longitudes for Mādhava's epoch as well as for present times.

Key words: Anomalistic cycle, Dhruva, Kalidina, Khaṇḍa, Mādhava, Moon, *Veṇvāroha.*

INTRODUCTION

On account of historical reasons the originality in Hindu astronomical writings suffered a setback in northern India since thirteenth century AD. But there had been uninterrupted development thereof in remote corners like Kerala, which were far removed from the political turmoils of northern India as is evidenced by the work of K.V.Sarma who has traced out more than 1000 texts of Kerala origin on astronomy and mathematics and enabled the identification of more than 150 astronomers of the land. Sarma's D.Litt thesis[1] entitled "Contributions to the Study of the Kerala School of Hindu Astronomy and Mathematics" extending over 2034 pages is a magnum opus on history of science that incorporated 14 very important astronomical-cum-mathematical treatises of the period from tenth to nineteenth

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century AD. As has been observed by Sarma[2] in the aforementioned work astronomical and mathematical development in India of the period since Bhāskara II is not well known and many important works that attests the presence of many great astronomers and mathematicians of Kerala remain little known in the field of history of science. One such great astronomer and mathematicaian was Mādhava of Sangamāgrāma who wrote the work *Veṇvāroha* for the computation of true longitude of Moon, a study of which forms the content of the present paper. Mādhava was one of the great geniuses of the times before Kepler and Newton and he lived at Irññgālakhusḍa (Saṇgama-grāma : 10°22' N, 76°15' E, of Kerala in the fourteenth-fifteenth century A.D. and had been the Guru of Parameśvara of *Dṛggaṇita* fame. Sarma has identified him to be the author of the following works.[3]

1. *Golavāda*
2. *Madhyamānayanaprakāra*
3. *Lagnaprakarana*
4. *Venvāroha*
5. *Sphuṭacandrāpti* and
6. *Agaṇita-grahacāra*

Among these *Golavāda* and *Madhyamānayanaprakāra* are known only through references in other works and only *Veṇvāroha* and *Sphuṭacandrāpti* are available in print. The rest is available in manuscript form. Mādhava's outstanding contributions to mathematics have come to light through the *uddhāraṇis* by the scholars who have come later in the tradition. Sarma has referred to two important anticipations of modern mathematical formulae by Mādhava, viz. the Taylor series approximation and Newton's Power Series for sine and cosine functions. A detailed account of the Taylor series approximation vis-à-vis Mādhava's formulation by R.C.Gupta[4] is available at page 92 of the *Indian Journal of History of Science*, May-Nov. 1969. In contrast to the mathematical contributions, the merits of Mādhava's astronomical works remain to be explored and the present paper is an effort in this direction. We have very little or almost no information about Mādhava's innovations in spherical astronomy vis-a-vis computation of planetary positions as the work *Golavāda* is not available. Both *Veṇvāroha* and *Sphuṭacandrāpti* attempt the computation of the true longitude of moon by making use of the true motions rather than the epicyclic astronomy of the Āryabhaṭan traditon. As such these texts offer us limited scope only to understand the ingenuity that a great mathematician like Mādhava's capability as an astronomer as well as mathematician. The present study makes use of *Veṇvāroha*[5] (along with the Malayālam commentary by Acyuta Pisārati) critically edited with Introduction and appendix by K.V. Sarma. *Veṇvāroha*, the title, literally means 'Ascending the Bamboo' and as shown by Sarma this title is reflective of the moon's computation in 9 steps over a day. This treatise makes use of the anomalistic revolutions for computing the true moon using the successive true daily velocity of moon framed in *vākyas* for easy memorization and use.

THEORY BEHIND THE INGENIOUS USE OF ANOMALISTIC CYCLES

An anomalistic cycle consists of 27 days and $13^h18^m34^s.45$. Therefore nine anomalistic cycles from a zero epoch will end respectively at:

Table 1

Cycle	Days	h-	m-	s
1	27	13-	18-	34.45
2	55	02-	37-	08.90
3	82	15-	55-	43.35
4	110	05-	14-	17.79
5	137	18-	32-	52.24
6	165	07-	51-	26.69
7	192	21-	10-	01.14
8	220	10-	28-	35.59
9	247	23-	47-	10.04

The nine cycles thus constitute nearly 248 days and the difference in longitudes of successive days ($\delta\lambda$) constitues the *candravākyas*. *Candravākyas* of Mādhava available in references (1) and (5) are reproduced in decimal notation as Appendix-1 of this paper. The series of *vākyas* begins from the moment when moon is at apogee and each *vākya* corresponds to the successive days' longitude of moon for the moment at which the anomaly was zero. It is apparent from column 3 that the end points of successive cycles render 9 gradations over a day. If we consider the cycles from a day at which moon's anomaly was zero at sunrise, the nine gradations shall become fixed moments at which moon's anomaly had been zero in a previous cycle. *Vākyas* could then be applied from that day to find the moon for the gradations like $13^h18^m34^s.45$ from sunrise.

1. Algorithm of Mādhava

The algorithm employed by Mādhava can be expressed in following steps:

a) Computation of 9 Dhruvas (D1, D2, ...D9) using Mādhava's Constants and Kalidina [K]

• [K - 1502008]*6845 ÷ 188611 → *agrimaphalam* [A] and balance [B_1], where **A** is the integral number of anomalistic revolutions.

Table 2

B_1	÷	6845	→ V_1	&	B_2
(B_2 + 188611)	÷	6845	→ V_2	&	B_3
(B_3 + 188611)	÷	6845	→ V_3	&	B_4
(B_4 + 188611)	÷	6845	→ V_4	&	B_5
(B_5 + 188611)	÷	6845	→ V_5	&	B_6
(B_6 + 188611)	÷	6845	→ V_6	&	B_7
(B_7 + 188611)	÷	6845	→ V_7	&	B_8
(B_8 + 188611)	÷	6845	→ V_8	&	B_9
(B_9 + 188611)	÷	6845	→ V_9		

• V_1, $\Sigma(V_1, V_2)$, $\Sigma(V_1, V_2, V_3)$, $\Sigma(V_1, V_2, V_3, V_4)$, $\Sigma(V_1, V_2, V_3, V_4, V_5)$, $\Sigma(V_1, V_2, V_3, V_4, V_5, V_6)$, $\Sigma(V_1, V_2, V_3, V_4, V_5, V_6, V_7)$, $\Sigma(V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8)$, and $\Sigma(V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9)$ become the 9 *vākya* numbers (Σ_1 to Σ_9). Illustration is provided in Table 3.

K = 1642675 : [1642675- 1502008]*6845/188611 → *agrimaphalam* **A** and balance **B_1** =6460

Table 3

						B_1=6460	
6460	÷	6845	→	V_1=0	&	B_2=3411	Σ_1=0
(3411 +188611)	÷	6845	→	V_2=28	&	B_3=362	Σ_2=28
(362 +188611)	÷	6845	→	V_3=28	&	B_4=4158	Σ_3=56
(4158 +188611)	÷	6845	→	V_4=27	&	B_5=1109	Σ_4=83
(1109 +188611)	÷	6845	→	V_5=28	&	B_6=4905	Σ_5=111
(4905 +188611)	÷	6845	→	V_6=27	&	B_7=1856	Σ_6=138
(1856 +188611)	÷	6845	→	V_7=28	&	B_8=5652	Σ_7=166
(5652 +188611)	÷	6845	→	V_8=27	&	B_9=2603	Σ_8=193
(2603 +188611)	÷	6845	→	V_9=28	&		Σ_9=221

• *Dhruvakālas* t_l = (6845 - B_l) ÷ 6845 days and *dhruvas*

For A = 5105 Mādhava give *dhruva* as 174°27' and for the next 69 anomalistic revolutions the arc is 211°44' followed by unit cycles of 03°04' 6".956. Illustration is given in Table 4.

Table 4

1	2	3	4	5	6
B_l	t_l in days	t_l in *ghaṭikās*	*dhruva* for A	*vākyas*	*dhruvas* D_l
6460	0.056245	3.374726	λ	δλ of Σ_1	Col.4+ Col.5
3411	0.50168	30.1008	λ - (i - 1)* 3.0686	δλ of Σ_2	Col.4+ Col.5
362	0.947115	56.82688	λ - (i - 1)* 3.0686	δλ of Σ_3	Col.4+ Col.5
4158	0.392549	23.55296	λ - (i - 1)* 3.0686	δλ of Σ_4	Col.4+ Col.5
1109	0.837984	50.27904	λ - (i - 1)* 3.0686	δλ of Σ_5	Col.4+ Col.5
4905	0.283419	17.00511	λ - (i - 1)* 3.0686	δλ of Σ_6	Col.4+ Col.5
1856	0.728853	43.73119	λ - (i - 1)* 3.0686	δλ of Σ_7	Col.4+ Col.5
5652	0.174288	10.45727	λ - (i - 1)* 3.0686	δλ of Σ_8	Col.4+ Col.5
2603	0.619722	37.18335	λ - (i - 1)* 3.0686	δλ of Σ_9	Col.4+ Col.5

b) Sorting t_l of column-3 and D_l of column-6 in the ascending order gives the *true longitudes for the respective moments represented by Dhruvakālas on the previous day* (*Iṣṭapūrvadina*) as the first *vākya* is zero. For *Iṣṭadina* the first *vākyas* is given.

c) Computed Results of Kalidina = 1642675 [13 July 1396 AD. mean sunrise 06.06 Ujjain Mean Time: JD (ZT): 22311140.7541666]

Table 5

1	2		3		4	5
Kalidina	*Dhruvakālas* & Moon		*Vākyas* No.	& $\delta\lambda$ $\delta\lambda$	9 *dhruvas* at expiry of cycles, *Iṣṭapūvadina*	9 *dhruvas* for the (+) 1 date
	03.3747	174°47'	0	0	174.783333	186.833333
	30.1008	171°43'	28	8°26'	180.148064	192.198064
	56.8269	168°39'	56	16°52'	185.512802	197.562802
	23.5530	165°35'	83	13°15'	178.827536	190.877536
1642675	50.2790	162°31'	111	21°41'	184.192267	196.242267
	17.0051	159°26'	138	18°05'	177.523639	189.573639
	43.7312	156°22'	166	26°31'	182.871740	194.921740
	10.4573	153°18'	193	22°54'	176.203141	188.253141
	37.1834	150°14'	221	31°20'	181.567875	193.617875

d) Comparison with Modern Longitudes

In making a contrast of the sidereal longitudes of Mādhava with the results of modern lunar theory we need to determine the *ayanāṃśa* of the epochs of which computations are presented. Mādhava has not spoken explicitly of the *ayanāṃśa* of his epoch and therefore we have to follow the Siddhāntic tradition of Kerala, which takes Kali 3623 or AD 522 as the coincidence of the vernal equinox with the zero point. The extra-long solar year of Siddhāntic astronomy necessitated a rate of precession of nearly one minute per year and this amounts to one degree in sixty years. *Ayanāṃśa* of any year Y can therefore be computed as (Y-522)/60 degrees. On this basis results of Mādhava are compared with modern longitudes for the day of expiry of anomalistic cycle, viz. 12 july 1396 A.D. at times of *Dhruvakālas* in Table 6 and for *Kalidina* of 1642675 or 13 July 1396 A.D. in Table 7.

The error has increased by about 10 minutes, as the precise variation in Moon's longitude is 11°53' instead of 12° 05', which the *vākya* (1) applies.

Table 6

1	2	3	4	5	6
Dhruvakālas in *gh-vigh*		3+ *ayanāṃśa*	Precise Julian dates for the moment:col.4 of Table.5	Modern true λ	Col. 3 -Col. 5
3.3747	174.7833	189.3503	2231139.810412	189°.00918	20'
10.45727	176.2031	190.7701	2231139.928454	190°.41153	22'
17.00511	177.5236	192.0906	2231140.037585	191°.70782	23'
23.55296	178.8275	193.3945	2231140.146716	193°.00401	23'
30.1008	180.1481	194.7151	2231140.255847	194°.30019	25'
37.18335	181.5679	196.1349	2231140.373889	195°.70231	26'
43.73119	182.8717	197.4387	2231140.483020	196°.99879	26'
50.27904	184.1923	198.7593	2231140.592151	198°.29556	28'
56.82688	185.5128	200.0798	2231140.701281	199°.59272	29'

Table 7

1	2	3	4	5	6
Dhruvakālas in *gh-vigh*	*Dhruvas*	2+*ayanāṃśa*	Precise Julian dates for the moment	Modern true λ	Col. 3 -5 Col. 5
3.3747	186.8333	201.3993	2231140.810412	200°.89036	31'
10.45727	188.2531	202.8191	2231140.928454	202°.29464	31'
17.00511	189.5736	204.1396	2231141.037585	203°.59364	33'
23.55296	190.8775	205.4435	2231141.146716	204°.89344	33'
30.1008	192.1981	206.7641	2231141.255847	206°.19414	34'
37.18335	193.6179	208.1839	2231141.373889	207°.60218	35'
43.73119	194.9217	209.4877	2231141.483020	208°.90508	35'
50.27904	196.2423	210.8083	2231141.592151	210°.20920	36'
56.82688	197.5628	212.1288	2231141.701281	211°.51462	37'

2. In terms of the Anomalistic Period of 27.5546538

Multipliers and divisors used in *Sphuṭacandrāpti* can be dispensed off in favour of the anomalistic period and the computation can be made more simple and algebraic as:

Let N be the number of days elapsed since the epoch at which anomaly was zero at sunrise. Number of anomalistic cycles (**c**) contained in **N** be **a** and the balance of days be **b**. Expiry of the a cycles will mark the moment or *dhruva* from which the *vākya* will be applied. The decimal part of **a**, (**a**-1), to (**a**-8) cycles [**c** = 27.5546538] converted to hours or *nāḍikās*, shall successively be the gradations over the day. For these gradations the *vākya* number can be obtained as **b**, (**b**+**c**), (**b**+2**c**), (**b**+3**c**)... (**b**+8**c**). The respective *dhruvas* can be found by using the cumulative arcs given: **a** = 5105: 174°27' and for the next 69 the arc is 211°44' followed by unit cycles of 03°04'6".956. Sum of the *dhruvas* and *vākyas* give the true moon correct for the meridian of Ujjain. *Dhruvakālas* arranged in the ascending order completes the process. The prescribed corrections can be applied for adoption to other meridians.

Example: True Moon for Kalidina

For Kalidina of 1625680 or Julian date = 2214145.75416 :01 January 1350 AD, 06:06 Ujjain mean time we get:

- **N** = 123672: **a** to (**a**-8) : 4488 to 4480
- *Dhruva* for **a** = 4488: 91°52' 21"
- *Vākya* numbers as: 7,34,62,89,117,144,172,199,227
- Tables 8 & 9 depict the results of computation as per *Venvāroha* compared to true Moon computed as per modern algorithms:

Table 8

1	2	3	4	5
Dhruvakāla	Julian date	Moon	*Vākya* No. & δλ	*Dhruva*
00-27	2214145.761628	70°23'32"	199(97°09')	167°33'
07-00	2214145.870759	76°31'46"	144(92°26')	168°58'
13-33	2214145.979889	82°40'00"	89(87°43')	170°23'
20-06	2214146.089020	88°48'14"	34(83°00')	171°48'
27-10	2214146.207062	67°19'25"	227(106°02')	173°21'
33-43	2214146.316193	73°27'39"	172(101°19')	174°47'
40-16	2214146.425324	79°35'53"	117(96°37')	176°13'
46-49	2214146.534455	85°44'07"	62 (91°55')	177°39'
53-22	2214146.643585	91°52'21"	7 (87°13')	179°05'

Table 9

1	2	3	4	5
Dhruvakāla in sequence	Julian date	Modern λ of Moon	Mādhava's Moon ÷ *Ayanāṃ* (13°48')	Difference Col.3-Col.4
00-27	2214145.761628	181°48'	181°21'	27'
07-00	2214145.870759	183°13'	182°46'	27'
13-33	2214145.979889	184°37'	184°11'	26'
20-06	2214146.089020	186°02'	185°36'	26'
27-10	2214146.207062	187°09'	187°09'	25'
33-43	2214146.316193	188°59'	188°35'	24'
40-16	2214146.425324	190°24'	190°01'	23'
46-49	2214146.534455	191°50'	191°27'	23'
53-22	2214146.643585	193°16'	192°53'	23'

The difference is nearly 25 minutes of arc only when compared with the results of latest algorithm of mathematical astronomy. It must be noted here that the date is not one of anomalistic conjunction where the technique is expected to give maximum accuracy. This example worked out using the multipliers and divisors of Mādhava gives the results:

Table 9

	B_i	V_i	*Vākyas*	*Dhruvakālas*: t_i	a	Fraction (a*c)*60
	48672	7	7	-6.11059		
	757	27	34	53.3645	4488	53.3644
Kalidina =	4553	28	62	20.0906	4487	20.0905
1625680	1504	27	89	46.8167	4486	46.8166
Khaṇḍaśeṣa =	5300	28	117	13.5427	4485	13.5427
123672	2251	27	144	40.2688	4484	40.2688
A=4488	6047	28	172	6.9949	4483	6.9948
	2998	27	199	33.7210	4482	33.7209
	6794	28	227	0.4470	4481	0.4470
	3745	28	255 or 7	27.1731	4480	27.1731

Dhruvakāla from the first balance (-) 6.11059 is the number of days at which the anomalistic cycle was complete. For *Kalidina* of 1625680. 7 days have passed in the new cycle and thus the *vākya* 7 corrects *dhruva* of **a** = 4488 to the true longitude of Moon.

3. Adapting Venvāroha for Recent Times

Mādhava's methodology, which he chose to describe as *Veṇvāroha*, can be applied to present times in a similar manner. Sidereal computation of Mādhava has to be dispensed with due to the difficulty in reconciling the siddhāntic and modern methods for computing *ayanāṃśa*. Epoch or the basement on which the *Veṇu* is planted needs to be modified to avoid cumulative error and also we shall use the tropical longitude of Moon in this exercise.

- Moon's anomalistic conjunction at Ujjain Mean sunrise [06:06: Zonal Time: ZT]
  →06 June 1951, 06:06 Ujjain Mean Time, JD [ZT] = 2433803.75434.
  Kalidina = 1845338 of *Dhruva* = 89°.27704427.

  →24 March 1984, 06:02 Ujjain Mean Time, JD [ZT] = 2445844.751117.
  Kalidina = 1857379 of *Dhruva* = 347°.273803.

  → 01 February 2000, 06:06 ZT, JD [ZT] = 2451575.75375.
  KD = 1863110[*Kalikhaṇḍa*]. *Dhruva* for this epoch = 262°.068271.

  → 17 April 2001, 06:06 ZT, JD [ZT] = 2452016.7543054.
  KD = 1863551[*Kalikhaṇḍa*]. *Dhruva* for this epoch = 312°.24746374.

Either of these epochs can be used as *dhruvadina or kalikhaṇḍa*. Modern Anomalistic period (**c**) of 27.5545486 days is very close to the value used by Mādhava, *viz.* 188611/6845 = 27.5546538 and as such the multiplier 6845 and divisor 188611 used by Mādhava can be retained.

Illustrations:

1. Kalidina = 1863551 which corresponds to Tuesday, 17 April 2001.

Dhruvadina is 1845338 where in Moon was at apogee at Ujjain mean sunrise. Wednesday, 6th June 1951. *Dhruvasphuṭa* = 89° 2770443. *Khaṇḍaśeṣa* = 18213, which gives *dhruva* of 314°.552311 and A = 660, b = 26.9871 and other computed results given in Table 10 and 11.

Table 10

Dhruvakāla	*Vākyas*	*Vākya* $\delta\lambda$	9- *Dhruvas*	*Dhruvakāla* in sequence	λ in sequence
0.7740	26	344°.32567	314°.552311	0.774	298°.87787
27.5001	54	352°.7775	311°.48371	7.8567	301°.62543
54.2262	82	1°.216666	308°.41511	14.4045	302°.94346
20.9523	109	357°.5969	305°.34651	20.9523	304°.26121
47.6784	137	6°.038889	302°.27792	27.5001	305°.67252
14.4045	164	2°.416111	299°.20932	34.5828	307°.00127
41.1306	192	10°.86056	296°.14072	41.1306	308°.3168
7.8567	219	19°.29528	293°.07212	47.6784	309°.63178
34.5828	247	15°.669	290°.00352	54.2262	312°.3674

Table 11

λ + 12°.5	Julian Date	Modern λs	Difference
310°.92787	2445844.768981	312°.3982	1°.47
313°.67543	2445844.878111	313°.7925	0°.12
314°.99346	2445844.987241	315°.0818	0°.09
316°.31121	2445845.105286	316°.3715	0°.06
317°.72252	2445845.214416	317°.6617	-0°.06
319°.05127	2445845.323546	319°.058	0°.01
320°.36680	2445845.432676	320°.3497	-0°.02
321°.68178	2445845.541806	321°.6422	-0°.04
324°.41740	2445845.659851	322°.9357	-1°.48

2. Kalidina = 1884857, which corresponds to Sunday, 17th August 2059 A.D.

Dhruvadina is 1845338 where in Moon was at apogee at Ujjain mean sunrise. Wednesday, 6th June 1951. *Dhruvasphuṭa* = 89°2770443. *Khaṇḍaśeṣa = 39519,* which gives *dhruva* of 169°.647851 and A = 1434, b = 26.9871 and other computed results given in Table 12 and 13

Table 12

1	2	3	4	5	6	7
Dhruvakāla	*Vākyas*	*Vākya* δλ	9- *Dhruvas*	Col.3+Col.4	*Dhruvakāla* in sequence	λ in sequence
14.7726	5	61°.62139	169°.64785	231°.26924	1.6770	228°.20346
41.4987	33	70°.06972	166°.57925	236°.64897	8.2248	229°.58482
8.2248	60	66°.07417	163°.51065	229°.58482	14.7726	231°.26924
34.9509	88	74°.81083	160°.44205	235°.25288	21.8553	232°.4696
1.6770	115	70°.83000	157°.37346	228°.20346	28.4031	233°.85986
28.4031	143	79°.55500	154°.30486	233°.85986	34.9509	235°.25289
55.1292	171	88°.32833	151°.23626	239°.56459	41.4987	236°.64897
21.8553	198	84°.30194	148°.16766	232°.46960	48.5814	238°.16239
48.5814	226	93°.06333	145°.09906	238°.16239	55.1292	239°.56419

Table 13

λ + 12°.5	Julian dates	Modern λs	Difference
240°.25346	2473322.782117	241°.01873	0°.77
241°.63482	2473322.891247	242°.40537	0°.77
243°.31924	2473323.000377	243°.79664	0°.48
244°.51960	2473323.118422	245°.30688	0°.79
245°.90986	2473323.227552	246°.70810	0°.80
247°.30289	2473323.336682	248°.11424	0°.81
248°.69897	2473323.445812	249°.52541	0°.83
250°.21239	2473323.563857	251°.05762	0°.85
251°.61459	2473323.672987	252°.47953	0°.86

3. Kalidina = 1874479, Thursday, March 20, 2031 AD

Dhruva of apogee conjunction of Moon: 1863551: λ = 312°.24746374 [17 April 2001]

Table 14

1	2	3	4	5	6	7
Dhruvakāla	*Vākyas*	*Vākya* δλ	9- *Dhruvas*	Col. 3 +Col. 4	*Dhruvakāla* in sequence	λ in sequence
36.4644	16	213°.600000	87°.412624	301°.012624	3.1905	293°.46069
3. 1905	43	209°.116667	84°.344025	293°.460692	10.2732	294°.76383
29.9166	71	218°.550000	81°.275426	299°.825426	16.8210	296°.53436
56.6427	99	227°.533333	78°.206827	305°.740160	23.3688	298°.53823
23.3688	126	223°.400000	75°.138228	298°.538228	29.9166	299°.82543
50.0949	154	232°.783333	72°.069629	304°.852963	36.4644	301°.01262
16.8210	181	227°.533333	69°.00103	296°.534364	43.5471	302°.8491
43.5471	209	236°.916667	65°.932432	302°.849098	50.0949	304°.85296
10.2732	236	231°.900000	62°.863833	294°.763833	56.6427	305°.74016

Table 15

λ + 12°.5	Julian dates	Modern λs	Difference
305°.51069	2462944.807342	305°.51390	0°.00
306°.81483	2462944.925387	306°.19254	0°.38
308°.58436	2462945.034517	308°.74408	0°.16
310°.58823	2462945.143647	310°.29522	0°.29
311°.87543	2462945.252777	311°.84587	0°.03
313°.06262	2462945.361907	313°.39594	0°.33
314°.89910	2462945.479952	315°.07190	0°.17
316°.90296	2462945.589082	316°.62049	0°.28
317°.79016	2462945.698212	318°.16822	0°.38

4. K = 1857379, Middle of the Anomalistic Period

Table 16

1	2	3	4	5
JD	*Vākya + Dhruva*	λ =Col.2+12°.5	Modern λ	Difference
2445858.768981	171°.3076	168°.476499	171°.30760	2°.831101
2445858.878111	172°.862	170°.347030	172°.86200	2°.51497
2445858.987241	174°.417	171°.267561	174°.41700	3°.149439
2445859.096371	175°.971	173°.221426	175°.97104	2°.749617
2445859.214416	177°.652	175°.024567	177°.65238	2°.627816
2445859.323546	179°.207	176°.061764	179°.20659	3°.144826
2445859.432676	180°.761	178°.098962	180°.76060	2°.661634
2445859.541806	182°.314	179°.169493	182°.31434	3°.144851
2445859.659851	183°.995	186°.155968	183°.99466	-2°.1613

These examples are illustrative of the salient features of Mādhava's method. Perturbations of the Moon's orbit are apparent across the four examples given above. Uniform accuracy is not apparent even for dates of anomalistic conjunction. In the fourth example that falls at the middle of the anomalistic period the error approaches 3°.0 due to perturbation terms. But still for an epoch such as 1400 AD when there was no lunar theory of more than four terms, accuracy such as above by so simple a method is really marvellous.

ACCURACY OF EPOCHAL POSITIONS OF MĀDHAVA

Mādhava employs two major epochs, *viz.*, *Dinakhaṇḍa* of 1502008 and 1644740.649 for the computation of Moon and Mean Sun respectively. Bamboo stands footed on 1502008, which according to the treatise marks a conjunction of the moon with its Higher Apsis at mean sunrise of Ujjain. The latter, 1644740.649, is the epoch for the computation of mean sun. Astronomical analysis of these epochs are explored further below:

(a) Dinakhaṇḍa of 1502008

If we take the beginning of Kaliyuga as to have taken place on the mean sunrise of Friday 18.02.3102 BC at Ujjain (JD = 588465.75416667) the *Kalidina* of 1502008 shall find the following astronomical description:

(i)1011 AD, May 29, Tuesday, 06:06 (mean sunrise); JD =2090473.75416 for LMT of Ujjain (Zonal Time): (TT =2090473.56186949) (True sunrise at Ujjain is 05:11:06 LMT and for Irinngalakkud 05:35 LMT).

According to *Venvāroha*: Mean moon = 349°35'09"10" (Sidereal)

Mean apogee = 349°35'07"57"

Modern algorithms give: Mean moon = 357°44'54" (Tropical longitude)

Mean apogee = 357°43'34" (true apogee =356°31')

Ayanāṃśa for Mādhava's scheme would have been [(1011-522/60] = 8°.15. Therefore Mādhava's tropical mean longitudes would have been: Moon = 357°44'09" and Apogee = 357°44'. Note the striking accuracy of Mādhava's mean moon and moon's apogee with that of the latest modern algorithms.

(ii)True conjunction of Moon and apogee took place for JD (TT): 2090473.4332761, Tuesday, May 29,1011 AD, 03:00:48.96 Local time of Ujjain (JD: 2090473.6256661). Moon's true longitude was 356°31'. On comparing these values with those of *Venvāroha*, it becomes apparent that the choice of the epoch, *Kalidina* = 1502008, superbly fitted the assumption of moon's conjunction with apogee even though it was 400 years earlier to his times and therefore could not have been observed. It is possible that the epoch may be of traditional origin and Mādhava might have only adopted it for his treatise.[6]

(b) Epoch of Mean Sun

Epoch of mean sun according to the text is, *Kalidina* of 1502008 + 5180 anomalistic revolutions of sun [5180 (18861/6845) = 5180*27.554565538 = 142732.6487] = 1644740.649 *Kalidina*. This will correspond to Ujjain sunrise on 10th March 1402 AD, (06:06) JD(ZT = Zonal time = Local Mean Time of Ujjain) = 2233206.75420778. (TT = 2233206.54819792).

Modern algorithm gives Sāyana Mean sun = 355°45'28"

Venvāroha gives Sidereal Mean Sun = 341°05'11" (6)

Ayanāṃśa of Mādhava = [(1402-522/60] = 14°40'

Mādhava's Sāyana sun = [341°05'11" + 14°40'] = 355°45'11"

This data is illustrative of the extreme accuracy of Mādhava in computing the mean sun.

(c) Data of Moon

Venvāroha contains two other epochs for which the moon's true longitude has been given:

1. [1502008 + 5174 anomalistic revolutions) = (1502008+142567.3213) = 1644575.3213 Kalidina = JD(ZT) = 2233041.07546; (TT = 2233040.86964056).

This corresponds to Sunday 25th September 1401 AD, 13:48:39.74 Ujjain LMT.

True *Sāyana* moon = 40°41'45"

Ayanāṃśa = [(1401-522)/60] = 14°39'.

Sidereal Moon = 26°02'45" (modern)

Mādhava's value for true moon = 26°31'

In this case the *sāyana* mean longitude of moon is 41°11'14" and the sidereal value of mean moon is 26°32'14".

2.[(1502008 + 5105 anom. rev (=140666.056245)] = 1642674.05624 *kalidinam*. This corresponds to 12th July 1396: 07:26:58.56, JD (ZT) = 2231139.8104:

True *Sāyana* moon = 189°00'33"; *Ayanāṃśa* = 14°34'

True sidereal moon = 174°26'33"

Mādhava's moon = 174°47'

The true longitudes of Moon given by Mādhava are in remarkable agreement with those of modern algorithms. In this case the sāyana mean moon is 189°21'51" and the sidereal mean moon is 174°47'51". In both the above cases, which mark the apogee conjunctions of moon. Mādhava's moon is exactly the same as the modern mean moon accounted for *ayanāṃśa*. It must be noted here that the modern mean moon has been computed as per the latest polynomial expression (8) $L = 218.3164591 + 481267.881342T - 0.0013268T^2 + 0.000001855835T^3 - 0.00001533883\,T^4$ where $T = [(JD\,(TDT) - 2451545)/36525]$. In the absence of this 4th order polynomial and the time correction δT (=400 seconds for 1400 AD) for the secular variation in earth's rotation perhaps we would not have been able to appreciate the accuracy of Mādhava. The true moon has been computed as per the Lunar solution ELP2000-82B for the apparent geocentric coordinate referred to the date ecliptic.

Comparison of the True Velocities

Mādhava has adopted 10th March 1402 Ujjayini mean sunrise 06:06 as the epoch for mean sun used in correcting the moon. The Table 17 gives the successive true longitudes of moon (λ), moon's speed in a day δλ, as well as δλ computed from Mādhava's *candravākyas* beginning with *śīlam rājñaḥ śriye* which are provided in decimal notation at Appendix - 1.

Table 17

Date Time 06:06	Longitude (λ) ELP2000-82B	δλ	δλ, Mādhava *Vākya*
March	51°10'39"	11°54'18"	———
09	63°03'23"	11°51'49"	12°02'04"
10	74°55'41"	11°53'30"	12°12'54"
11	86°51'54"	11°59'42"	12°22'43"
12	98°56'38"	12°10'32"	12°35'01"
13	111°14'29"	12°25'53"	12°49'11"
14	123°49'44"	12°45'15"	13°04'31"
15	136°46'01"	13°07'43"	13°20'07"
16	150°05'45"	13°31'53"	13°35'15"
17	163°49'44"	13°55'54"	13°49'04"
18	177°56'45"	14°17'32"	14°03'53"
19	192°23'15"	14°34'33"	14°10'03"
20	207°03'40"	14°45'05"	14°16'10"
21	221°50'54"	14°48'07"	14°23'54"
22	236°37'23"	14°43'41"	14°13'05"
23	251°16'10"	14°32'56"	14°13'51"
24	265°41'45"	14°17'39"	14°06'20"
25	279°50'39"	13°59'53"	13°55'54"
26	293°41'18"	13°41'26"	13°43'07"
27	307°13'45"	13°23'39"	13°28'38"
28	320°29'05"	13°07'19"	13°13'11"
29	333°28'58"	12°52'44"	12°57'35"
30	346°15'09"	12°39'55"	12°42'42"
31	358°49'19"	12°28'40"	12°29'16"
April	11°12'55"	12°18'45"	12°18'05"
1	23°27'14"	12°10'04"	12°09'23"
2	35°33'29"	12°02'40"	12°04'05"
3	47°33'03"	11°56'46"	12°02'09"
4	59°27'39"	11°52'48"	12°03'43"
5	71°19'29"	11°51'19"	12°08'43"
6	83°11'19"	11°52'56"	———
7	95°06'35"	11°58'16"	———

It is apparent from the above comparison that the profile of modern true motion of moon per day over an anomalistic revolution is in agreement with that of Mādhava's Vākyas.

Use of Anomalistic Revolutions

Mādhava's computations stems from 1011 AD, May 29, Tuesday, 06:06 (mean sunrise), JD = 2090473.75416. This matched well with the moon's transit over apogee on May 29, 03:00:48.96 LMT of Ujjain (2233206.75420778), as can be expected from the use of anomalistic revolutions, very nearly coincided the apogee transit of moon for JD of 2233206.73153396 corresponding to Friday, 12th March, 1402 AD, 05:33:24.53 LMT of Ujjain. The other two epochs depict the following features:

(i) 25th September 1401 AD, 13:48:39.74 Ujjain LMT, JD = 2233041.07546; Apogee transit: 25th September 1401 23:42:28.82 LMT (Ujjain), JD = 2233041.48783354.

(ii) 12th July 1396, 07:26:58.56, Ujjain LMT, JD (ZT) = 2231139.8104; Apogee transit: 12 July, 1396, 10:24:36.18 LMT (Ujjain), JD =2231139. 9337526.

Mādhava's choice of the expiry of 5180 anomalistic revolutions at sunrise of 12th March 1402 AD as the epoch was closer to the apogee of moon than others, i.e., 5105 or 5174 revolutions.

Apogee of Sun

Mādhava gives: solar apogee $= 78°$, perigee $= 258°$

$1°$ motion for $\lambda = 327°03'10''$ and $188°56'50''$

In the epochal year of 1402 AD, modern astronomy gives apogee at sāyana 93°01'32" (sidereal position will be 78°21'32") on 16th June 22:54. Sun had 1° motion on 17th February 1402 1200 LMT for $\lambda = 337°02'46''$ (sidereally 332°22'46") and 13 October 1402, 1200 LMT for $\lambda = 208°07'51''$ (sidereally 193°28'). It is evident that the apogee is very accurate but the solar positions for 1° daily motion has got a displacement by (+) 5° and (-)5° respectively.

Sphuṭacandrāpti

Sphuṭacandrāpti [7] contains the same method of computation of moon as well as candravākyas except for some minor details. But one of the appendices to the edition contains valuable data of Mādhava's true moon longitudes given as the *dhruva* for different lumps of *kalidina*:

Table 18

1	2	3	4	5	6
Kalidinam	Date : UMT	Julian Date	Sidereal λ,	Tropical λ,	Modern λ,
1645705	29.10.1404,Wed.06:06	2234170.54828	151°37'34"	166°19'34"	166°55'40"
1633333	15.12.1370,Sun.06:06	221798.54909	213°49'24"	227°57'24"	228°36'33"
1620961	31.01.1337,Thu.06:06	2209426.54997	276°01'14"	289°35'14"	290°08'45"
1608589	18.03.1303,Mon.06:06	2197054.55092	338°13'04"	351°14'04"	351°19'06"
1521985	06.02.1066,Mon.06:06	2110450.55944	53°35'56"	62°39'56"	62°43'25"

The Table 18 gives Mādhava's sidereal longitudes in column (4) which have been converted to the respective tropical values of column (5) using *ayanāṃśa* of 14°42', 14°08', 13°35', 13°01' and 09°04' for AD 1404, 1307, 1337, 1303 and 1066 respectively. Among these five epochs the last two results of Mādhava differ from the ELP2000-82B[8] value only by (-) 5 minutes of arc as the mean sunrise of Ujjain very nearly coincided with moon's apogee transit (18.03.1303, 02:19 and 06.02.1066, 05:35 respectively). Even in the first three cases Mādhava's longitudes differ by roughly (-) 40 minutes of arc only. The remarkable agreement for distant epochs such as 1303 AD and 1066 AD is suggestive of the observational origin of the data.

CONCLUSIONS

The above analysis conveys the realization that not only Mādhava's method was simple but also the most accurate one. Even under a comparison with the modern computer derived longitudes Mādhava's accuracy is amazing and this is true not only about moon but also of other parameters such as mean sun, apogee conjunction of moon etc. Of course, Mādhava was one of the greatest mathematicians of his times and as such this achievement is not surprising. Sarma has referred to Mādhava's appellation as *golavid* and in the light of the above analysis it appears very appropriate. Mādhava's genius reflected in *Veṇvāroha* is an indirect pointer towards the sound astronomical tradition of Kerala that pre-dates even Aryabhata. *Vakya* computation of moon can be traced back to the times of Vararuci, who lived around 3rd -4th century AD.[9]

APPENDIX- I

śīlaṃ rājñaśriyetyādi candravākyas of Mādhava

1.	12.043056	31.	44.918611	61.	78.867222
2.	24.144167	32.	57.384167	62.	91.919889
3.	36.359167	33.	70.069722	63.	105.221111
4.	48.737778	34.	83.001667	64.	118.781667
5.	61.321389	35.	96.193056	65.	132.575833
6.	74.141111	36.	109.642500	66.	146.571111
7.	87.216389	37.	123.335833	67.	160.724444
8.	100.551667	38.	137.246389	68.	174.985278
9.	114.139167	39.	151.335000	69.	189.298056
10.	127.956944	40.	165.554722	70.	203.603889
11.	141.971667	41.	179.851389	71.	217.845278
12.	156.139167	42.	194.167500	72.	231.966667
13.	170.408611	43.	208.444722	73.	245.919722
14.	184.723611	44.	222.626111	74.	259.663333
15.	199.025000	45.	236.659722	75.	273.167500
16.	213.255833	46.	250.500833	76.	286.415556
17.	227.361389	47.	264.114722	77.	299.403889
18.	241.293056	48.	277.478333	78.	312.141389
19.	255.011667	49.	290.581667	79.	324.652222
20.	268.488889	50.	303.428889	80.	336.970833
21.	281.708611	51.	316.036944	81.	349.141667
22.	294.668333	52.	328.435833	82.	1.216667
23.	307.380000	53.	340.666389	83.	13.253056
24.	319.867778	54.	352.777500	84.	25.309167
25.	332.169167	55.	4.823889	85.	37.442778
26.	344.325556	56.	16.864167	86.	49.707500
27.	356.393611	57.	28.956389	87.	62.150833
28.	8.429444	58.	41.156389	88.	74.810833
29.	20.491389	59.	53.514722	89.	87.715278
30.	32.636667	60.	66.074167	90.	100.877778

91. 114.299722
92. 127.967778
93. 141.856111
94. 155.927500
95. 170.135556
96. 184.426389
97. 198.743333
98. 213.027500
99. 227.221944
100. 241.273889
101. 255.138056
102. 268.777778
103. 282.169722
104. 295.301389
105. 308.176111
106. 320.808889
107. 333.228889
108. 345.475556
109. 357.596944
110. 9.648056
111. 21.685833
112. 33.769444
113. 45.955278
114. 58.294167
115. 70.830000
116. 83.596111
117. 96.614722
118. 109.893333
119. 123.427500
120. 137.197500
121. 151.172500
122. 165.310833
123. 179.562778
124. 193872500
125. 208.182222
126. 222.433056
127. 236.570278
128. 250.543611
129. 264.311667
130. 277.843611
131. 291.120000
132. 304.136111
133. 316.900000
134. 329.433889
135. 341.771389
136. 353.955833
137. 6.038889
138. 18.076667
139. 30.128056
140. 42.250278
141. 54.498333
142. 66.920000
143. 79.555000
144. 92.431667
145. 105.565833
146. 118.960000
147. 132.601944
148. 146.467778
149. 160.521389
150. 174.716944
151. 189.001667
152. 203.318611
153. 217.608889
154. 231.816389
155. 245.885833
156. 259.772500
157. 273.438333
158. 286.858056
159. 300.018333
160. 312.920278
161. 325.578333
162. 338.020000
163. 350.283333
164. 2.416111
165. 14.471667
166. 26.508333
167. 38.584167
168. 50.755833
169. 63.075833
170. 75.588333
171. 88.328333
172. 101.318611
173. 114.568889
174. 128.070278
175. 141.821389
176. 155.775833
177. 169.898611
178. 184.140833
179. 198.446944
180. 212.759444
181. 227.008889
182. 241.171944
183. 255.165278
184. 268.957222
185. 282.515833
186. 295.820833
187. 308.865278
188. 321.656111
189. 334.213889
190. 346.570556
191. 358.769167
192. 10.860556
193. 22.900556
194. 34.947500
195. 47.056667

196. 59.291111
197. 71.691944
198. 84.301944
199. 97.151389
200. 110.257222
201. 123.623056
202. 137.239167
203. 151.082222
204. 165.117222
205. 179.299444
206. 193.577222
207. 207.893611
208. 222.189722
209. 236.408611
210. 250.495833
211. 264.404444
212. 278.095833
213. 291.543056
214. 304.732222
215. 317.661667
216. 330.345556
217. 342.808889
218. 355.089444
219. 7.233889
220. 19.295278
221. 31.331111
222. 43.399722
223. 55.558611
224. 67.860000
225. 80.349722
226. 93.063333
227. 106.025556
228. 119.247500
229. 132.726944
230. 146.447500
231. 160.381111
232. 24.487778
233. 188.719444
234. 208.023056
235. 217.336111
236. 231.605000
237. 245.771389
238. 259.784444
239. 273.600278
240. 287.185556
241. 300.518611
242. 313.590556
243. 326.409167
244. 338.990833
245. 351.367778
246. 3.581667
247. 15.681944
248. 27.724722

Notes and References

1. Sarma, K.V., *Contributions to the Study of the Kerala School of Hindu Astronomy and Mathematics*, VVRI, Hoshiarpur, 1977. See also his later work, *Science texts in Sanskrit in the manuscript repositories of Kerala and Tamilnadu* - A documented survey which identifies 2420 texts and above 1500 authors. Rashtriya Sanskrit Sansthan, New Delhi 2002.
2. Ibid. Ch. IV. p 11
3. Ibid. Ch. II. p 71
4. Gupta, R.C. 'Second Order Interpolation in Indian Mathematics up to the Fifteenth century', IJHS, 4.1-2 (1969) 93.
5. *Veṇvāroha* ed by K.V.Sarma, Sri Ravi Varma Sanskrit series no. 7, The Sanskrit College Committee, Tripunithura, 1956.
6. *Vākya* computation of moon, which makes use of the anomalistic revolutions, pre-dates even Āryabhaṭa and the originator of the method Vararuci supposedly belonged to the 3rd /4th century AD. So it is quite likely that the epoch of *Kalidina* = 1502008 at which the moon conjoined apogee at sunrise is of traditional origin rather than arising from back computation by Mādhava. It is quite unlikely that the back computation over 400 years would have given so much accuracy for the moon's apogee conjunction.
7. Critically Edited and translated , Introduction and Appendices by K.V.Sarma, Visheshvaranand Institute, Hoshiarpur, 1973. See ref. 1, 4.1255
8. Lunar solution ELP2000-82B, Bureau des longitudes, Paris
9. K.V.Sarma has given the Kali chronograms, which according to the tradition marked the birth and death of his son Melattol Agnihotri as 1257921 and 1270701 respectively. These chronograms correspond to Wednesday, 18th February 343 AD and 14th February 378 AD, Friday, respectively. On the latter epoch at mean sunrise of Ujjain, moon was having an almost perfect conjunction with the perigee and can therefore be an ancient astronomical chronogram employed in the *vākya* process of Vararuci. But as there is no history of the use of perigee in the history of Hindu astronomy, this may be an accident.