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ARITHMETIC OPERATION OF DIVISION WITH SPECIAL REFEFENCE TO BHĀSKARA II'S LĪLĀVATĪ AND ITS COMMENTARIES

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The Sanskrit terminology for division reveals the two fold nature of division. From the earliest known times, Indians had a sound knowledge of the concept and they succeeded in establishing certain methods for dividing one number by another. Two such methods are enunciated in Bhāskara II s *Līlāvatī*. Various commentators on the *Līlāvatī* have thrown more light on it. The method of division by removal of common factors and the long division method using common numerals, widely used by Indians from very early times came into general use in the outside world only after several centuries. The most simple inverted divisor method of division of fractions given by Bhāskara II date back to 499 AD, if not earlier, but it is found that it came into general use in the European countries only in the 16th and 17th centuries AD. With the advancement of knowledge of the concept of division, ancient Indians brilliantly succeeded in giving a meaning and terminology for the expression of the form a/0 and the remarks of Bhāskara II and some of the commentators reveal the nature of such a quantity, thereby introducing the abstract concept of infinity. Moreover, an analysis of the views expressed by Bhāskara II regarding the division of a *khaguṇa* by *kham* reveal the conception of *kham* (*Śūnya*) as infinitesimal thereby introducing the concept of limits.

Keywords : Galley Method, *Khahara, Kriyakramakari*, Līlāvatī, *Mahāsiddhānta.*

Division is the process of finding the missing factor in multiplication when one factor of the product and the product are given. The number which corresponds to the product in multiplication is called *bhājya or hārya* (dividend) and the known factor of the product is called *bhajāka* or *hāra* (divisor). The unknown factor sought which is obtained after performing the operation of division is called *labdhi* or *labdha* (quotient). Any number left over undivided is called *śeṣa* (remainder). The process of division is called *bhāgahāra* or *bhājana* or *haraṇa* or *chedana,* meaning 'to break into parts', 'to partition', 'to take away' etc. The Sanskrit terminology clearly reveals the two fold nature of division, one

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based on partitioning or breaking up of the dividend and the other based on the repeated subtraction or taking away of a number of equal units from the dividend such that the whole formed by the units taken together along with the remainder left over if any will be the dividend. From this it is clear that the division operation has been related to multiplication as well as to subtraction. Just as multiplication can be regarded as a case of repeated addition with all the addends alike, division can be regarded as a case of repeated subtraction with all the subtrahends alike. Just as subtraction is the inverse of addition operation, division is the inverse of the multiplication operation. This has been explicitly mentioned by Śankara in his commentary *Kriyākramakarī* (1534 AD) on Bhāskara's *Līlāvatī* (1150 AD). According to Śankara, '*Anena guṇana parikarma viparītarūpatvam haraṇasya darśitam*' [1]. Division is seen to be the inverse of the multiplication operation,. The same idea has been conveyed by Gaṇeśa in his commentary *Buddhivilāsini* (1545 AD) on the *Līlāvatī* [2]. Damodar Misra describes the process of division as the case of repeated subtraction also in his '*Vāsanā*' [3] on the *Līlāvatī.*

Even though division was not mentioned as a fundamental arithmetical operation in the Vedic literature, certain references pertaining to division can be had from the Vedic literature. Division of one thousand into three equal parts has been spelled out explicitly in the Ṛgveda [4]. Division of 1000 by 3 is mentioned in the *Taittirīya Saṃhitā* also.

> *"ye twin have conquered; ye are not conquered;*
> *neither of the two of them has been defeated.*
> *Indra and Vishnu, when ye contended,*
> *ye did divide the thousand into three'* [5].

In the *Śatapatha Brāhmaṇa* [6], this division is mentioned clearly giving 333x3+1. *The full explanation of the arithmetic feat of Indra and Viṣṇu in the proposed division is given* in the *Śatapatha Brāhmaṇa* as follows: *for when Indra and Viṣṇu divided a thousand (cows) into three parts, there was one left, and here they caused to propagate herself in three kinds; and hence, even now, if anyone were to divide a thousand by three, one would remain over* [7]. Moreover since fractions are inevitably expressions of division and the knowledge of fractions can be traced as far back as the earliest of the Vedas,[8] the knowledge of division can also be traced as far back as that period. Also division of 10 by 5 giving the quotient 2, division of 10 by 2 giving the quotient 5, division of 100 by 5 giving 20 and that of 100 by 2 giving 50 etc are stated in the *Śatapatha Brāhmaṇa*[9]. Again the division of 720 by 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24 are given which give the quotients 360, 3 x 80, 180, 144, 120, 90, 80, 72, 60, 48, 45, 40, 36, 30 respectively without leaving any remainder and the division of 720 by 7, 11, 13, 14, 17, 19, 21, 22 & 23 are left out because 720 is not completely divisible by them

[10]. Thus a fair knowledge of the concept of divisibility of one number by another is evident from this.

While the process of division was considered to be difficult and tedious by the Europeans even upto 16th cent A.D [11], it was not so for the ancient Hindus who used the process extensively while extracting square roots and cube roots. Division process is also involved in the rule of three, rule of five etc, in the determinations pertaining to the mixture of things and in the formulae for summation and successive summation of natural numbers etc. Even though the division process plays a dominant role in the methods of extraction of square and cube roots given by Āryabhaṭa I and Brahmagupta, they have not included the process of division in their works. This non inclusion of the process of division may be due to the fact that the process was well known and considered too elementary to be included in their Siddhānta works [12].

Regarding the process of division, Bhāskara II enunciates in the most celeberated work Līlāvatī as follows:

> *"bhājyādharaḥ śudhyati yadguṇaḥ syād,*
> *antyāt phalam tatkhalu bhāgahāre*
> *samena kenāpyapavartya bhājyahārau*
> *bhajedvā sati sambhave tu"* [13].
>
> *'That number, by which the divisor being multiplied balances the last digit of the dividend (and so on) is the quotient in the division, or, if practicable, first abridge both the divisor and the dividend by an equal number, and proceed to division'* [14].

The process has been illustrated by dividing 1620 by 12. Various commentators on the Līlāvatī have given detailed exposition of the process with varying degree of clarifications. Gaṇeśa remarks that the rationale of the process by abridging the divisor and dividend by some common number follows obviously from the reverse of *rūpaguṇa* form of multiplication [2]. Śankara has cited earlier references of the process of division in the *Kriyākramakarī.*

The method of division given by Bhāskara II by removing common factors (*apavartanam*) was common in late middle ages under the name '*Per Rapiego*' [15]. But the knowledge of this method in India can be traced as far back as 150 B.C and can be found in the *Tatvārthādigama sūtra - bhāṣya* of Umāsvāti [16]. The method also appears in the works of Mahāvīra [17] and Srīdhara [18] prior to Bhāskara's *Līlāvatī.*

Of the methods which make use of our common numerals in long division, Gerbert's method attributed to Gerbert of 980 A.D is stated to be one of the oldest form although it is uncertain whether he originated it and although he did not use the zero [19]. But the method has been used extensively by the Indians prior to 980 A.D. The earliest reference of the method can be had from Srīdhara's *Triśatika* of 750 A.D [20]. The method also appears in Mahāvīra's *Gaṇitasāra Saṃgraha* of 850 A.D [21] and in *Mahāsiddhānta* of *Āryabhaṭa* II of 950 A.D [22].

While Bhāskara II describes the process briefly in the *Līlāvatī*, more details of the process are provided by the various commentators on it. In the *Manoranjana* of 15th cent AD, an exposition of the rule has been given by Rāmakṛṣṇa Deva using Bhāskara's illustrative example of dividing 1620 by 12. "*The highest places of the proposed dividend, 16, being divided by 12, the quotient is 1 and 4 over. Then 42 becomes the highest remaining number, which when divided by 12 gives the quotient 3, to be placed in a line with the preceeding quotient 1; thus getting 13. Remaining 60, which when divided by 12 gives 5; and this being carried to the same line as before, the entire quotient is exhibited: viz. 135*" [23]. This is exactly the modern plan. In the commentary *Gaṇitāmṛtasāgari* of 1420 AD, Gangādhara states that the divisor is to be repeated for every digit of the dividend like the multiplier in multiplication [24]. This is exactly the distributive principle which is the main principle for modern algorism. '*By adding the several numbers and dividing their sum by the given number, the quotient is the sum of quotients obtained by dividing each of the several numbers by the given number*' [25]. This principle $\frac{x+y}{d} = \frac{x}{d} + \frac{y}{d}$ is applied for dividing 'a' by 'b' by splitting the dividend 'a' (*bhājya*) according to the places of figures and repeating the divisor 'b' (*bhājaka*) for every digit of the dividend. Thus according to Gangādhara's remark, if $a = d_n 10^n + d_{n-1} 10^{n-1} + \ldots\ldots\ldots\ldots + d_0 10^0$ then $\frac{a}{b} = \frac{d_n}{b} 10^n + \frac{d_{n-1}}{b} 10^{n-1} + \ldots\ldots\ldots + \frac{d_0}{b} 10^0$ where the divisor b is repeated for every digit of the dividend.

Now, for two integers x, y, ∃ two integers q and r are such that $x = qy + r$ where $0 \le r < y$ with

$$
\begin{aligned}
q &= 0 \text{ if } x < y \text{ and} \\
&= \text{an integer if } x > y \\
\text{and } r &= 0 \text{ if } x \text{ is divisible by } y.
\end{aligned}
$$

Thus $\frac{d_n}{b} = q_n + \frac{r_n}{b}$ so that

$$
\begin{aligned}
\frac{a}{b} &= \left(q_n + \frac{r_n}{b}\right) 10^n + \frac{d_{n-1}}{b} 10^{n-1} + \ldots\ldots\ldots \frac{d_0}{b} 10^0 \\
&= q_n 10^n + \left(\frac{r_n 10 + d_{n-1}}{b}\right) 10^{n-1} + \frac{d_{n-2}}{b} 10^{n-2} + \ldots\ldots + \frac{d_0}{b} 10^0 \\
&= q_n 10^n + \left(q_{n-1} + \frac{r_{n-1}}{b}\right) 10^{n-1} + \frac{d_{n-2}}{b} 10^{n-2} + \ldots\ldots + \frac{d_0}{b} 10^0
\end{aligned}
$$

$$(\text{where } r_n 10 + d_{n-1} = b\, q_{n-1} + r_{n-1})$$

$$= q_n 10^n + q_{n-1} 10^{n-1} + \left(\frac{r_{n-1} 10 + d_{n-2}}{b}\right) 10^{n-2} + \ldots\ldots + \frac{d_0}{b} 10^0$$

Proceeding like this, $a/b = q_n 10^n + q_{n-1} 10^{n-1} + \dots\dots\dots + q_0 10^0 + (r_0/b)$ where each q_i is some digit from 0 to 9 obtained as the quotient of division of $r_{i+1} 10 + d_i$ by b and each r_i is the remainder left over if any, where i = n, n-1, ...1,0 and $r_{n+1} = 0$. This corresponds to the modern practice of division if the divisor is written on the left of the dividend and the digits q_n q_{n-1} etc, called partial quotients, are written above the corresponding digits of the dividend and the successive steps are written down one below the other.

The long division method which is probably of Indian origin, was transmitted to Europe as 'Scratch method' or 'Galley method' through the Arabs in the 9th cent A.D and became popular in Europe from 15th to 18th cent A.D [26]. The modern method of long division derived from this came into being as early as the 15th cent A.D with some variations in the position of both the quotient and the divisor [27]. In the *Kriyākramakarī*, the positions of quotient and divisor are described in almost the same manner as used in modern plan.

Regarding the division of fractions, Bhāskara II states:

> *"chedam lavam ca parivartya harasya śeṣaḥ*
> *karyoatha bhāga haraṇe guṇanāvidhiśca"* [28]

"Having interchanged the numerator and denominator of the divisor fraction, the remaining process for division of fractions is the same as that in multiplication [29].

Earlier reference of this rule can be had from the *Āryabhaṭīya* [30], *Brahmaspuṭa siddhānta* [31], *Pāṭīgaṇita* [32] and these are cited in the *Kriyākramakarī*. Earlier to Bhāskara II, the rule has been stated by Mahāvīra [33], Āryabhaṭa II [34] & Srīpati [35,36] also. The rationale of the rule has been given by both L. Jha [37] and Damodar Misra [38] in their respective commentaries on the *Līlāvatī*. If x = a/b is the dividend fraction, y = p/q is the divisor fraction, then, a = x.b; p = y.q and so dividend fraction x inverted divisor fraction = (a/b). (q/p)

$$= (x.b/b).\ (q/y.q)$$

$$= \frac{x}{y} = \frac{\text{dividend fraction}}{\text{divisor fraction}}$$

Here the fractions x.b/b and q/y.q are multiplied to get x/y after applying the process of reduction called '*apavartana*'. The *apavartana* process was known to the Indians from atleast 150 BC [16]. The process was known to Āryabhaṭa I as he has mentioned the inverted divisor method of division of fractions which is in a way based on the cancellation technique. The method of division of fractions using inverted divisor given by Indian arithmeticians had made the most difficult operation of division of fractions very simple. In this context the following remarks made by D.E. Smith are worth mentioning.

> *Naturally the most difficult operation was division. Multiplication by the inverted divisor is so simple that we hardly realize that it has come into general use only recently, although it was known in the early Middle Ages by the Hindus and the Arabs* [39].

Among the Arabs *Al Hassar (of 1175 A.D?) recognized it, atleast with integral dividends and among the Hindus, Brahmagupta and Bhaskara gave it* [39]. Again, as remarked by D.E Smith.

> *It is curious that this device used in the East became a lost art until again adopted in Europe in the 16th cent A.* [40].

This simple method which seems to have dropped out of sight of the outside world for several centuries, reappeared in Stiefel's works in 1544 A.D, and was not at once accepted, but it became fairly common and came into general use only in 17th cent AD[42]. But this method was well known and commonly used by the Indian arithmeticians from the very ancient times, as early as 499 AD, if not earlier.

Now pertaining to division of a definite quantity by zero, the quotient of such division has been termed *khacheda* by Brahmagupta and *khahara* by Bhāskara II. According to Bhāskara II, *khabhājito rāśiḥ khaharaḥ syāt* [42] which means that a quantity having zero divisor is a *khahara.* According to the commentator Gaṇeśa [43] *khahara* is such a large unlimited quantity that it cannot be determined how great it is, which means that it is an infinite quantity. Also *khahara* ± k = *khahara,* which is unaltered, where k is any definte quantity. Gaṇeśa refers to ācārya's enunciation viz; '*asmin* vikāraḥ *khahare na rāśiriti*' [43] which means that this *khahara* is a changeless quantity. Regarding the nature of *khahara, Bhāskara has made his own remark in Prījagaṇita* that in *khahara there is no alteration whatsoever may be inserted or extracted as no change takes place in the infinite and immutable god, at the period of destruction or creation of worlds though numerous orders of beings are absorbed or put forth* [44]. According to the commentator Rangnātha of 1620 AD in his *Vāsanābhāṣya, the quantity* is infinite because smaller the divisor is, the greater is the quotient; now cipher being in the utmost degree small, given a quotient infinitely great.[45] In this context the following remark of G.H. Lewis is worth mentioning.

> *'The infinity is no more a quantity than zero is a quantity. If zero is a sign of vanished quantity. infinity in a sign of that continuity of existence'* [46].

Incidentally, it may be observed that while the European mathematicians like Martin Ohm of 1828 A.D viewed a/0 as having no meaning when $a \neq 0$ (for the quotient x0 gives only 0 and not a as long as $a \neq 0$) [47], Bhāskara II of 1150 A.D succeeded in giving a meaning, terminology and even the nature of the expression a/0. Moreover the commentators like Gaṇeśa of 1545 AD, Śankara of 1534 AD, and Ranganātha of 1620 AD have given their explanations regarding the nature of such an expression in their respective commentaries.

Finally regarding the division of a multiple of 'zero' (*khaguṇa*) by 'zero' Bhāskara states *śūnye guṇaka jate kham harascet punasthadā rāśiḥ avikṛta eva jñeya* [48] which means that *śūnya* having become a multiplier and *kham* a divisor, the quantity is understood to be unaltered. Regarding *khaguṇa*, Bhāskara states *"khaguṇaḥ kham, khaguṇaścintyaśca śeṣa vidhau"* [49], which means that a multiple of 'zero' is zero but should be retained as such (*khaguṇa*), if further operations impend. With this statement Bhāskara introduces the abstract idea that kx *śūnya* need not be śūnya but it must be retained as such if further operations impend, but otherwise, kx śūnya = śūnya. From this it is clear that the so called *śūnya* is conceived of here as an *infinitesimal* which ultimately reduces to zero. Applying this principle, Bhāskara proceeds to state that *śūnya* / kham = k where k remains unaltered or it is *avikṛta*. From this also it may be inferred that the terms *kham* and *śūnya* used by Bhāskara in this context are conceived of as limit of a diminishing quantity δ which ultimately tends to zero. This hidden idea of the conception of the terms *śūnya* or *kham* as infinitesimal $\delta \to 0$ by Bhāskara has been made more evident by Śankara in *Kriyākramakarī* [50]. Thus Bhāskara may be said to have introduced the concept of limits and Bhāskara's observations on this when written in modern notation of limits emerge as follows :

$$\lim_{\delta \to 0} \frac{k.\delta}{\delta} = k$$

The following illustrative example given by Bhāskara plainly justifies the above inference regarding Bhāskara's knowledge of the concept of limits: "what is the quantity whose product with '*kham*' added to half (of the product), then (the whole) multiplied by three and divided by '*kham*' gives 63" [51]. The answer given is 14. ie find k such that $\frac{(k.0 + k.0/2)^3}{0} = 63$ where the notation 0 is used to denote '*kham*'. With this 0 notation for *kham* it may look absurd since 0/0 is indeterminate in usual notation when 0 stands for absolute zero. But the answer 14 given by Bhāskara suggests that the '*kham*' mentioned here is conceived of as an infinitesimal δ which is as close to zero as possible, but not as absolute zero. In modern notation using limits the above example may be put in the form; find k,

$$\lim_{\delta \to 0} \frac{(k.\delta + k.\delta/2)^3}{\delta} = 63 \; ?$$

Evaluating the limit and solving the resulting equation, the value of k is 14 which is exactly the answer given by Bhāskara.

Thus while the concept of division of a definite quantity by zero leads to the abstract concept of *khahara*, the infinity; the division of a multiple of infinitesimal 'zero' by the infinitesimal leads to the concept of limits and Bhāskara seems to be the first to introduce this concept. Bhāskara's remark that the operations with zero is of great use in astronomical computations has been justified by Śankara in *Kriyākramakarī* with the help of an illustrative astronomical example [52].

The above discussion throws light on the two fold nature of division (revealed by the division terminology itself as well as by certain commentators on the *Līlāvat),* and on the knowledge of the concept of division that the Vedic sages had. It also throws light on the advancement of this knowledge by the ancient Indian arithmeticians from the earliest known times to Bhāskara II and then by the commentators on the Līlāvatī of Bhāskara like Gangādhara, Śankara & Nārāyaṇa, Gaṇeśa, Ranganātha etc. The ancient Indians succeeded in establishing certain methods for dividing one number by another number and in introducing the concepts of infinity and limits motivated from the division by zero. Of the two methods of division given by Bhāskara, the method by removing common factors (*apavartanam*) date back to 150 B.C. This method became common in the outside world only in the late middle ages under the name *Per Rapiego.* The other method of Bhāskara resembling the modern method is based on the distributive principle and it probably dates back, to the days of Aryabhata I (499 AD), even though Śridhara (750 AD) stands first in including the method in his work. The distributive principle used in this process which is the main principle for modern algorism, has been clearly revealed by Gangādhara of 1420 AD. The inverted divisor method of division of fractions was commonly used by the Indian arithmeticians from 499 AD, if not earlier, but in the outside world, this simple method came into general use only in the 17th cent AD. While some of the European arithmeticians viewed a/0 as meaningless, Indian arithmeticians succeeded in giving a meaning and terminology for such an expression and the remarks of Bhāskara and the commentators like Gaṇeśa, Śankara, and Raṅganātha reveal the nature of such a quantity, thus introducing the abstract concept of infinity. Bhāskara also succeeded in floating the concept of limits by treating the so called *śūnya* or *kham* as infinitesimals in dealing with multiple of *kham* and in the division of a multiple of *śūnya* by *kham* and this shows the richness of tradition of ancient Indian mathematics.

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43. *Līlāvatī* of Bhāskarācārya with *Buddhivilāsini & Līlāvatīvivaraṇa* of Gaṇeśa & Mahīdhara, Part I, Ed. by D.V. Apte, Poona, (p 39), *ibid* 2.

44. History of Hindu Mathematics Part I by B.B. Datta and A.N. Singh (p 243), ibid 12.

45. Algebra with Arithmetic & Mensuration from Sanskrit of Brahmagupta & Bhāskara by H.T. Colebrooke (p 19, footnote 5), *ibid* 14.

46. Positive Sciences in the Vedas by D.D. Mehta (p125), Arnold Heinemann Publishers, New Delhi, 1974.

47. Lehrbuchder niedern Analysis Vol I, Berlin, 1828 (p 110-112), also History of Hindu Mathematics part I by B.B. Datta & A.N. Singh (p 246, footnote 2), *ibid* 12.

48. Līlāvatī of Bhāskarācārya with Kriyākramakarī of Śankara & Nārāyaṇa Ed. by K.V. Sarma (p 91, verse 46), *ibid* 1.

49. *Kriyākramakarī* of Śankara & Nārāyaṇa Ed. by K.V. Sarma (p91, verse 45 .ii), *ibid* 1.

50. *Kriyākramakarī* of Śankara & Nārāyaṇa Ed. by K.V. Sarma (p 92-94), *ibid* 1.

51. *Kriyākramakarī* of Śankara & Nārāyaṇa Ed. by K.V. Sarma (p 92), *ibid* 1.

52. *Kriyākramakarī* of Śankara & Nārāyaṇa Ed. by K.V. Sarma (p 94), *ibid* 1.