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FIRST DEGREE INDETERMINATE ANALYSIS IN ANCIENT INDIA AND ITS APPLICATION BY VĪRASENA

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This paper provides an interpretation, through *kuṭṭaka* to Vīrasena's formula $c = 3d + (16d + 16) / 113$.

Key Words : *First degree indeterminate analysis; Kuṭṭaka; Saṃśliṣṭa kuṭṭaka.*

INTRODUCTION

The most general form of a linear equation in x and y ($x > o$, $y > o$) may be written as :

$$Lx + My + N = O \qquad \text{.......................} \quad (1)$$

Where $L > O$, M and N are integral constants.

Equation (1) assumes the following four forms :

$$ax + by + c = o \qquad \text{.....................} \quad (1.1)$$

$$ax - by + c = o \qquad \text{.....................} \quad (1.2)$$

$$ax + by - c = o \qquad \text{.....................} \quad (1.3)$$

$$ax - by - c = o \qquad \text{.....................} \quad (1.4)$$

Under the condition that a, b and c are all positive, the above equations reduce to the following two forms :

$$ax - by = + c \qquad \text{.....................} \quad (2)$$

$$ax + by = c \qquad \text{.....................} \quad (3)$$

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Assume that (a, b) = 1, i.e., a and b are relatively prime. If a and b have a common factor, then so must have c. Remove this comon factor such that

(coefficient of x, coefficient of y) = 1.

The credit of finding the solution of first degree indeterminate equation (2) by a method called *kuṭṭaka* (literal) meaning : pulverizer, i.e., to obtain the solution by breaking into smaller fragments by means of continued division), in the history of Indian mathematical sciences, goes to Āryabhaṭa I (b. 476 A. D.)[1]. This method of solution resembles the continued fraction process of Euler[2] given in 1764. The concept of first degree indeterminate analysis did not appear in India all of a sudden. It has some footing in the *Śulbasūtras,* in the form of simultaneous indeterminate equations, which originated due to the problems of altar construction[3]. After Āryabhaṭa I, the *kuṭṭaka* method[4] for (2) was subsequently discussed by Bhāskara I (ca. 625 A.D.), Brahmagupta (ca.628 A.D.), Govindasvāmī (ca. 800 A.D.), Mahāvīra (ca. 850 A.D.), Pṛthudakasvāmī (ca. 860 A.D.), Āryabhaṭa II (ca. 950 A.D.), Bhāskara II (b. 1114 A.D.), Devarāja (between 1200 and 1700 A.D.), Nārāyaṇa Paṇḍita (ca. 1356 A.D.), Jyeṣṭhadeva (ca. 1530 A.D.), Kamalākara (fl. 1616-1700 A.D.), Putumana Somayaji (ca. 1732 A.D.) and others. On the other hand, Brahmagupta[5] for the first time solved equation (3) by converting into the type ax-by = c. Note that the indeterminate equation (2) always possesses an infinite number of solutions whereas indeterminate equation (3) has a finite number of solutions or sometimes no solution[6]. Several scholars, such as Chasles[7], Colebrooke[8], Kaye[9], Smith[10] and Sen[11] regard various *kuṭṭaka* methods to be purely original to India and not influenced by any other ancient culture.

LINEAR SIMULTANEOUS INDETERMINATE EQUATIONS

To find a number N which when divided by given positive numbers $a_1, a_2, \ldots\ldots, a_n$ leaves the remainders (positive numbers) $r_1, r_2, \ldots\ldots, r_n$ respectively, i.e., to solve the (n-1) simultaneous equations.

$$N = a_1 x_1 + r_1 = a_2 x_2 + r_2 = \ldots\ldots\ldots = a_n x_n + r_n.$$

Remarkable that a common solution certainly exists if a_i's are pairwise coprime or G.C.M. of a_i, a_j divides $(r_i - r_j)$ for all i, j such that $1 \leq i \leq j \leq n$. Brahmagupta[12] obtains the solution of such equations by *kuṭṭaka*. Bhāskara I[13], Mahāvīra[14] and Bhāskara II[15] also discuss the similar problems through *kuṭṭaka*.

Mahāvīra[16] applies *kuṭṭaka* to solve simultaneous equations of a more specific type :

$b_1y = a_1x + c_1$
$b_2y = a_2x + c_2$
....
....
$b_ny = a_nx + c_n$

known as the problem of *Saṃśliṣṭa kuṭṭaka*, i.e., constant pulverizer. Subsequent mathematicians, namely, Āryabhaṭa II, Bhāskara II, and Devarāja follow Mahāvīra closely[17].

As *kuṭṭaka* had become an important tool of algebra till the time of Vīrasena (ca. 816 A.D.), it is quite likely that he possessed a deep knowledge of this kind of analysis. In this paper, we apply certain *kuṭṭaka* methods to solve Vīrasena's equation relating to the circumference of a circle and its diameter. This technique removes ambiguity about its existence and establishes a strange but interesting range for π.

Āryabhaṭa I's Value of π

Āryabhaṭa I (*Āryabhaṭīya* II. 10, c. 499 A.D.) states[18] the following rule to obtain surprisingly such a good value of π ;

caturādhikaṃ śatam aṣṭaguṇaṃ dvāṣaṣṭis tathā sahasrāṇām,
ayutadvayaviṣkambhasyāsanno vṛttapariṇāhaḥ

"One hundred and four, multiplied by eight, plus sixty-two thousand : (this is) the nearly approximate (length of the) circumference of a circle whose diameter is twenty thousand."

Āryabhaṭa I's rule can thus be expressed as :

$c_0 = (62832 / 20000)\ d_0$. .... (4)

This implies

$\pi = 62832 / 20000 = 3.1416.$

This value of π is a close approximant *(āsanna)* in the sense of Āryabhaṭa I. Bhāskara II uses it in the reduced form 3927/1250 and calls it accurate *(sūkṣma)*.

VĪRASENA'S APPLICATION OF KUṬṬAKA

Vīrasena in his commentary, called *Dhavalāṭīka* (c. 816 A.D.), on *Ṣakhaṇḍāgama* of Puṣpadanta and Bhūtabali-a post canonical work of the Jaina Digambara sect-quotes[19]:

"vyāsaṃ ṣoḍaśaguṇitaṃ ṣoḍaśasahitaṃ trirūparūpair bhaktam,
vyāsatriguṇitasahitaṃ sūkṣmād api tad bhavet sūkṣmam."

"The diameter multiplied by 16, increased by 16, divided by 113, and (again) combined with thrice the diameter is (the circumference) more accurate than the accurate one."

In modern form, this can be expressed as :

$$c = 3d + (16d + 16) / 113. \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

It is equivalent to

$$p = 355 / 113 + 16 / (113d) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Where $p = c / d$.

(6) is monotonically decreasing as :

$$P_{n+1} - P_n = (16 / 113) [(d_n - d_{n+1}) / (d_n d_{n+1})] < 0$$

for $d_n < d_{n+1}$.

From (5),

$$113c = 355d + 16.$$

Solving equation (7) by *kuṭṭaka*, we obtain the general solution as :

$c = 352 + 355t$; $d = 112 + 113t$, where $t = 0, 1, 2, \ldots\ldots\ldots$

So

π Corresponding to $t = 0$, i.e., $d = 112$ is

$$P_{112} = 352 / 112 = 22 / 7.$$

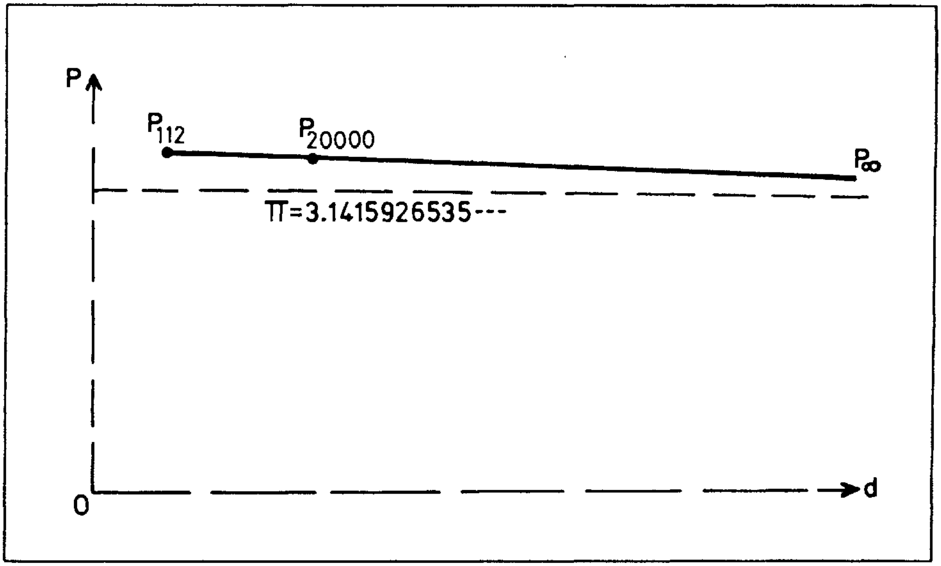
π Corresponding to t = 176, i.e., d = 2000 is

P2000 = 62832 / 20000.

π Corresponding to t = ∞ , i.e., d = ∞ is

P_∞ = 355 / 113.

Thus we may construct the following diagram :



.Conclusion

The method of of *kuṭṭaka* for solving(5) establishes the following facts :

(i) The strangeness of the formula that it is dimensionally unbalanced, as has been noticed by Gupta[20], Hayashi et a1,[21], Sarasvati[22], Singh [23] and others, has been removed.

(ii) It confirms the argument of Gupta[24] Hayashi et al[25] , that Vīrasena's formula considers Āryabhaṭa I's value to be a good ideal as it is deducible from equation (5).

(iii) Hayashi et al[26] have done a good work on the value of.

However, their conclusion[27] *"in India such an effort to find out range for the value of* π *seems to have never been made in ancient and medieval times"* is not acceptable. Indeed, from the above figure.

$355/113 < \pi < 22/7$.

Though this range does not include actual value of π accurate to more than 6 decimal places, yet it allures to peep into its peculiarty. However, a comparision may be made with the inequality of Archimedes (fl. 287-212 B.C.)

$223/71 < \Pi < 22/7$

(iv) The words quoted by Virasena *'more accuràte than the accurate one'* clearly signify the value of π more accurate than Aryabhata I's value which is obtained when $d > 20000$; and so

$355/113 < \pi < 62832/20000$, i.e.

$3.1415929....... < \pi < 3.1416$.

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