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INDETERMINATE ANALYSIS IN THE CONTEXT OF THE MAHĀSIDDHĀNTA OF ĀRYABHAṬA II

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The subject of indeterminate analysis has been very well researched by the modern researchers. The concentrated effort of Āryabhaṭa II in the field of indeterminate equations has also been estimated. But a general method for a solution of indeterminate equations given by Āryabhaṭa II was not expressly stated with illustrations. The object of the present paper is to throw light on a general method for the solutions of indeterminate equations applicable to all types of problems discussed in the *Mahāsiddhānta* of Āryabhaṭa II.

The scientific study of the sun, moon, stars etc. is, of course, closely connected with the subject of indeterminate analysis.[1] It is for this reason that the topic has been dealt with by most of the mathematicians of ancient and medieval periods.[2] It is noteworthy that the bifurcation of this branch of learning from the science of algebra, too, emphasizes its importance.[3]

Though Āryabhaṭa II confined to make concentrated effort[4] to improve the solution of indeterminate equations of the form

$b\ y = a\ x \pm c$; a, b, c being positive integers, his general as well as specific rules for the solutions of the problems is quite praise-worthy.

GENERAL RULE FOR THE SOLUTIONS IN POSITIVE INTEGERS OF THE EQUATION OF THE FORM $B\ Y = A\ X \pm C$; A, B, C BEING POSITIVE INTEGERS

Firstly, Āryabhaṭa II observes that there may be five operations in all and they are:

भाज्यक्षेपच्छेदा यथोदिताः संस्थिताः क–विधिरेषः।
ते च करण्या भक्ता दृढाभिधाना अयं ख–विधिः।।
भाज्यक्षेपौ ग–विधिः क्षेमच्छेदौ यदा तदा घ–विधिः।
भाज्यक्षेपौ क्षेपच्छेदौ ङ–विधिर्विभिन्नकरणीभ्याम्।।

Mahāsiddhānta, XVIII, 1, 2.

i.e. "Setting down the dividend, interpolator and divisor as stated in the problem is the first operation, to divide them by their greatest common divisor so as to make them without a common factor is the second operation; to divide the dividend and interpolator by their greatest common divisor is the third operation; to divide the interpolator and divisor by their greatest common divisor is the fourth operation; to divide the dividend and interpolator and then the reduced interpolator and divisor by their respective greatest common divisors is the fifth operation."

Here the statement for the first two operations is interpreted as step 1 of the general rule for the solutions of the problems which is as follows:

Step: 1. We divide the dividend (a), interpolator (c) and divisor (b) by their greatest common divisor, if any.

The process of finding two final *kuṭṭa* is described below:

भाज्यहरावन्योन्यं विभजेत् टा–शेषकं भवेद्यावत्।
सा वल्ली तेन हतेऽन्त्येनोर्ध्वे कान्विते स्फुटा वल्ली।।
विषमसमत्वं ज्ञात्वाऽनष्टोपान्त्येन ताडिते स्वोर्ध्वे।
स्वस्थानच्युतमन्त्यं योज्यमनेन प्रकारेण।।
राशी कुट्टाख्यौ स्तो वक्ष्येऽन्यौ तौ सदा विषमजाख्यौ।।
सकृदेवच्छेदहृते भाज्ये शेषं यदा टा स्यात्।।
लब्धं तदोर्ध्वकुट्टः शेषं चाधः स्थितो ज्ञेयः।

Mahāsiddhānta XVIII, 4-7 (a)

i.e. "The reduced dividend and the divisor are mutually divided until we get the remainder as unity. The quotients are placed one under the other successively to get an auxiliary chain. The ultimate is multiplied by the penultimate and then one is added. When the result obtained replaces the penultimate, it gives the correct chain. Knowing the number of quotients, whether even or odd, the undestroyed penultimate is multiplied by the number just above it and then the ultimate number is added. Rejecting the ultimate, we get a new chain. The process is repeated till a pair of numbers called *kuṭṭa*, is left. On dividing the dividend by the divisor once only, if we get the remainder as unity, the quotient obtained in this case is treated as the upper *kuṭṭa* and the remainder 1 as the lower *kuṭṭa*."

Thus, the step 2 for the solution may be stated as follows:

Step: 2. The numbers a and b are divided reciprocally until the remainder becomes unity. The quotients obtained are placed one below the other successively to form an initial chain. The ultimate quotient is multiplied by the penultimate quotient and then one is added to the resulting product. The chain is again formed after replacing the penultimate quotient by the result obtained. The new penultimate number is multiplied by the quotient just above it and the ultimate quotient is added to the product obtained. The number just above the new penultimate is replaced by the result leaving the ultimate quotient. The reduced chain is obtained and the process is continued till the chain reduces to two numbers. These two numbers are termed as the upper and the lower *kuṭṭa*. In case, a = b m + 1, m being a whole number, the quotient m is treated as the upper *kuṭṭa* and the remainder one as the lower *kuṭṭa*.

For further process of the solution, Āryabhaṭa II says:

कुट्टौ स्वक्षेपहतावूर्ध्वाधः स्थौ क्रमाद्भक्तौ।।
निजभाज्यच्छेदाभ्यां फलगुणकौ शेषकौ भवतः।

Mahāsiddhānta, XVIII 7(b), 8(a).

i.e. "The resulting upper and lower *kuṭṭa* are multiplied by the interpolator. The products obtained are divided by the divident and the divisor. The residues thus obtained will be the quotient and multiplier respectively."

The fragment of the aforesaid text may be treated as the step 3 which is as follows:

Step: 3. The upper *kuṭṭa* (u) and the lower *kuṭṭa* (1) are multiplied by c separately and the resulting products are divided by a and b respectively. The quotients obtained may either be equal or unequal.

It is relevant to note here that the step 4, which is mentioned below, is based on the following texts of Āryabhaṭa II.

एवमभीष्टविधिभवौ फलगुणकौ प्रस्फुटौ धनक्षेपे।
समवल्ल्यां विषमायामृणसंज्ञे क्षेपके स्याताम्।।
समवल्ल्यामृणसंज्ञे धनसंज्ञे वा विषमवल्ल्याम्।
स्वविधौ फलगुणहीनौ सुदृढौ भाज्यच्छिदौ फलगुणौ स्तः।।
अन्यत्र प्रश्नोक्तावथ तत्सम्वधजे यदा लब्धी।
न समे गुण एव तदा ग्राह्यो हेयं फलं धनक्षेपे।।
फलमृणसंज्ञे ग्राह्यं हेयो गुणको गुणात् फलोत्पत्तिम्।
गुणपृच्छाभाज्यवधं पृच्छाक्षेपेण संस्कृतं विभजेत्।
प्रश्नोक्तच्छेदेन स्पष्टं लब्धं फलं भवति।।
प्रश्नच्छित्फलघातं व्यस्ताख्यक्षेपकेण संस्कृत्य।
प्रश्नोदितेन पृच्छाभाज्येन भजेद् गुणो भवेल्लब्धम्।।

Mahāsiddhānta, XVIII, 13-16(a), 17, 18.

i.e. "In the case of positive interpolator, when the chain is even, and also in the case of negative interpolator, when the chain is odd, the quotient and multiplier are obtained correctly by the process mentioned above. Again, when the quotient and the multiplier obtained are subtracted respectively from the dividend the divisor made prime to each other, the remainders are the correct quotient and multiplier in the case of even chain and negative interpolator and also in the case of odd chain and positive interpolator. In a particular problem, where the interpolator is positive and the quotients obtained are unequal, the derived value for the multiplier is accepted rejecting that of the quotient. For the negative interpolator, the derived value of quotient is accepted and that of the multiplier is rejected. Multiplying the accepted value of the multiplier by the dividend of the proposed question and adding the interpolator, and then dividing the result by the divisor of the proposed question, the correct quotient is obtained. Again, multiplying the accepted quotient by the proposed divisor and adding the reverse of the interpolator, and then dividing the result by the dividend of the proposed question, we get the correct multiplier."

The study of the text quoted above impel us to categorize the problems and their solutions as stated in step 4 below:

Step: 4. In the fourth step, the problems reduce to the following categories:

(i) Equal quotients, even chain[5] and positive c

(ii) Equal quotients, odd chain and negative c

For both the categories of problems, the remainders obtained in step 3 become the required least positive integral values of y and x respectively.

(iii) Equal quotients, even chain and negative c

(iv) Equal quotients, odd chain and positive c

For these categories (iii) and (iv), the remainders obtained in step 3, when subtracted from a and b, give the least positive integral values of y and x respectively.

(v) Unequal quotients, even chain and positive c

The remainder obtained, on dividing the product of l and c by b, is the least integral value of x. Knowing the value of x, the value of y can be calculated with the help of the given equation.

(vi) Unequal quotients, odd chain and positive c

The remainder obtained, on dividing the product of l and c by b, when subtracted from b, gives the required value of x. By putting the value of x in the given equation, the required least positive value of y can be obtained.

(vii) Unequal quotients, even chain and negative c

The remainder obtained, on dividing the product of u and c by a, when subtracted from a, gives the required least integral value of y. Substituting the value of y in the given equation, the least integral value of x is calculated.

(viii) Unequal quotients, odd chain and negative c.

The remainder obtained, on dividing the product of u and c by a, gives the least integral value of y. The value of x is calculated by putting the value of y in the given equation.

It is further important to note that with regard to the general solutions of the problems, Āryabhaṭa II says:

फलगुणकौ युक्तौ स्तः प्रश्नोक्ताभ्यामभीष्टगुणिताभ्याम् ।
भाज्यच्छिद्भ्यां बहुधा सुदृढाभ्यां चेष्टगुणिताभ्याम् ।।

Mahāsiddhānta, XVIII, 20.

i.e. "The minimum quotient and multiplier being added respectively with the dividend and divisor as stated in the question or as reduced, after multiplying both by an optional number give various other values."

Thus, to explain the rule, we may write step 5 as follows:

Step: 5. The general solution in positive integers of the given equation is obtained with the help of the equations $x = b\,m + \alpha$, $y = a\,m + \beta$; $\alpha.\beta$ being least solution in positive integers of the given equation and m, a whole number.

It is now worth mentioning that the general rule stated is applicable to all problems of the form $b\,y = a\,x \pm c$; a, b, c being integers.

Moreover, Āryabhaṭa II gave some specific rules meant for different types of problems which are as follows:

Type: 1 When the dividend (a) and the interpolator (c) have a common factor

Type: 2 When the divisor (b) and the interpolator (c) have a common factor

Obviously, these are the third and the fourth operations as classified by Āryabhaṭa II. For these two types, Āryabhaṭa II proposes to divide (a, c) and (b, c) by their respective greatest common divisors [*MSi*, XVIII, 2(a)]

Further steps of the solution are discernible in the following couplets:

ग–विधावूर्ध्वं कुट्टं प्रश्नक्षेपेण संगुणयेत्।।
करणीजक्षेपेणाऽधः स्थं घ–विधावतो व्यस्तम्।
अनयोर्विध्योरेवं गुणितौ कुट्टौ क्रमाद्भक्तौ।।
पृच्छककथितविभाज्यच्छेदाभ्यां फलगुणौ शेषौ।

Mahāsiddhānta, XVIII, 8(b)-10(a).

i.e. "In case of the third operation, we multiply the upper *kuṭṭa* by the interpolator of the question and the lower *kuṭṭa* by the interpolator as reduced by dividing by the greatest common divisor. The same should be done reversely in the case of the fourth operation. For these two operations, the *kuṭṭa* after being multiplied, as indicated, when divided respectively by the dividend and the divisor stated by the questioner, the residues obtained will be the respective quotient and multiplier."

Thus the specific rules for these two types may be summerized as such:

Specific rule: 1. When a and c have common divisor

Dividing a and c by their greatest common divisor, we get the quotients as a_1 and c_1 respectively. Then a_1 and b are divided reciprocally until the remainder becomes unity. The remaining process of getting the two final *kuṭṭa* is adopted as explained in step 2 of the general rule. The upper *kuṭṭa* is multiplied by c and the lower *kuṭṭa* by c_1. The resulting products are divided by a and b respectively. The quotients obtained are either equal or unequal. The step 4 and 5 of the general rule are then followed to get the solution.

Specific rule: 2. When b and c have a common divisor.

Dividing b and c by their greatest common divisor, we get the quotients as b_1 and c_1 respectively. Then a and b_1 are processed as in step 2 of the general rule to get the two final *kuṭṭa*. The upper *kuṭṭa* u is multiplied by c_1 and the lower *kuṭṭa* 1 by c. The resulting products are divided by a and b respectively. The quotients obtained may either be equal or unequal. Then the steps 4 and 5 of the general rule are applied to get the required solution.

Type: 3. When (a, c) and (b, c) both have common divisors

This is the fifth operation of Āryabhaṭa II. In this case, he again proposes to divide the dividend and interpolator and then the interpolator thus reduced and divisor by their respective different greatest common divisor [*MSi*, XVIII, 2(b)]

To get the solution of such problems, he suggests:

भाज्यक्षेपकरण्या ङ–विधावूर्ध्वं तलस्थमन्यकया।।

हन्यान्मध्यफलगुणौ .प्रश्नच्छेदं फलेन संगुणयेत् ।
भाज्यं गुणकेन तथा सद्विवरं हार इष्टः स्यात् ।।
प्रश्नक्षेपघ्नौ फलगुणकौ मध्यावभीष्टहारहृतौ ।
लब्धी प्रश्नविभाज्यच्छेदहृते फलगुणौ शेषौ ।।

Mahāsiddhānta, 10(b)-12.

i.e. "In the fifth operation, we multiply the upper *kuṭṭa* by the greatest common divisor of the dividend and interpolator, and the lower one by the greatest common divisor of the given divisor and reduced interpolator. The products are the intermediate quotient and multiplier. Again, we multiply the divisor of the question by the intermediate quotient and also the dividend by the intermediate multiplier. The difference of these products is the required intermediate divider. The intermediate quotient and multiplier are multiplied by the interpolator of the question, and then divided by the intermediate divider. The quotients thus obtained when divided respectively by the dividend and divisor of the question, the residues obtained will be quotient and multiplier required."

The explanation of the rule stated above is designated as specific rule 3, which runs thus:

Specific rule: 3. When (a, c) and (b, c) both have common divisors.

Dividing a and c by their greatest common divisor d_1 (say), we get the respective quotients as a_1 and c_1. Again, dividing b and c_1 by their greatest common divisor d_2 (say), we get b_1 and c_2 as the quotients respectively. Now a_1 and b_1 are divided mutually till the remainder becomes unity. The step 2 of the general rule is then applied to get the two final kuṭṭa as u and 1.

Dividing $\frac{c\ d_1\ u}{b\ d_1\ u \sim a\ d_2\ 1}$ and $\frac{c\ d_2\ 1}{b\ d_1\ u \sim a\ d_2\ 1}$

by a and b respectively, we get the quotients, which are either equal or unequal. The steps 4 and 5 of the general rule are then followed to get the required solution.

Besides above specific rules, two more specific rules are also derived from the following śloka of Āryabhaṭa II.

स्वक्षेपे छेदहृते निरग्रके ना गुणः फलं लब्धिः ।
एवमृणक्षेपे नो ना–क्षेपे फलगुणौ नौ स्तः

Mahāsiddhānta, XVIII, 19.

i.e. "In case, where the interpolator being divided by the divisor yields zero as the remainder, the quotient obtained is the required quotient and zero, the multiplier. But in case of zero interpolator, the quotient and multipler are both zero."

Hence the specific rules 4 and 5 may be interpreted thus:

Specific rule: 4. When c is positive and divisible by b

The quotient obtained, when c is divided by b, is the required value of y, the value of x being zero.

Specific rule: 5. When c is zero

In this case $x = 0$, $y = 0$ is the required least solution.

Five examples are cited below for the illustrations of the rules stated. They are, of course, collected from other sources.[6]

Example: 1. Find the general solution of the equation $14x - 11y = 29$ in positive integers.[7]

Solution: We rewrite the given equation as

$$11\ y = 14\ x - 29$$

Here $a = 14$, $b = 11$, $c = 29$ and c is negative.

Applying the step 2 of the general rule, we get

1	1	5 (u)
3	4	4 (l)
1	1	

(Initial chain)

The step 3 of the general rule gives equal quotients each being equal to 10 and different remainders as 5 and 6 respectively.

Since the quotients obtained are equal, the chain is odd and c is negative, the problem reduces to the category 4(ii). Hence $y = 5$ and $x = 6$ is the least solution in positive integers.

Applying the step 5 of the general rule, we get the general solution in positive integers as x = 11 m + 6, y = 14 m + 5, m being a whole number.

Example: 2. Find the general solution of the equation 12x – 29y = 700 in positive integers.[8]

Solution The equation can be rewritten as

$$29\ y = 12\ x - 700$$

Here a = 12, b = 29, c = 700 and c is negative.

Applying the step 2 of the general rule, we get

0	0	0	5 (u)
2	2	12	12 (l)
2	5	5	
2	2		

(Initial chain)

By using the step 3 of the general rule, we get 291 and 289 as the unequal quotients and 8 and 19 as the remainders respectively.

Since the quotients are unequal, the chain is even and c is negative, the rule for category 4 (vii) gives y = 4, x = 68, the least solution in positive integers. The general solution in positive integers in this case is x = 29 m + 68, y = 12 m + 4, m being a whole number.

Alternative Solution: Here a and c have the greatest common divisor 4 and hence the problem can be solved with the help of the specific rule 1 also.

To begin with, a = 12 and c = 700 are divided by their greatest common divisor 4. The quotients thus obtained are a_1 = 3 and c_1 = 175 respectively. Now a_1 = 3 and b = 29 are divided reciprocally until the remainder comes to unity. The step 2 of the general rule gives

0	0	1 (u)
9	10	10 (l)
1	1	

(Initial chain)

Here u x c = 1 x 700 = 700, 1 x c_1 = 10 x 175 = 1750.

Dividing the products 700 and 1750 by a = 12 and b = 29, we get 58 and 60 as the quotients and 4 and 10 as the remainders respectively.

Since the quotients obtained are unequal, the chain is odd and c is negative, the problem comes under the category 4 (viii). Hence y = 4 and x = 68 is the required least solution in positive integers. The general solution is naturally the same as obtained by the general method.

Example: 3. Find the general solution of the equation 199 x – 225 y = 20 in positive integers.[9]

Solution: We rewrite the equation in the form

$$225\ y = 199\ x - 20$$

Here a = 199, b = 225, c = 20 and c is negative.

Applying the step 2 of the general rule, we get

0	0	0	0	0	23 (u)
1	1	1	1	26	26 (l)
7	7	7	23	23	
1	1	3	3		
1	2	2			
1	1				

(Initial chain)

The step 3 of the general rule gives 62 and 70 as the respective remainders, the quotients being each equal to 2.

Since the quotients are equal, the chain is even and c is negative, the problem comes under the category 4 (iii). Hence x = 155, y = 137 is the least solution in positive integers.

The application of the step 5 of the general rule gives x = 225 m + 155, y = 199 m + 137 as the required general solution in positive integers, m being a whole number.

Alternative Solution: Since b = 225 and c = 20 have the greatest common divisor 5, we apply the specific rule 2. So b = 225 and c = 20 are divided by their greatest common divisor 5. The quotients obtained are $b_1 = 45$ and $c_1 = 4$ respectively. Now a = 199 and $b_1 = 45$ are processed as in step 2 of the general rule to get the two final *kuṭṭa* as:

4	4	4	4	84 (u)
2	2	2	19	19 (l)
2	2	8	8	
1	3	3		
2	2			

(Initial chain)

Here $u \times c_1 = 84 \times 4 = 336$, $l \times c = 19 \times 20 = 380$.

Dividing the products 336 and 380 by a = 199 and b = 225, we get the remainders as 137 and 155 respectively, each of the equal quotients being 1. Since the quotients are equal, the chain is odd and c negative, the problem comes under the category 4 (ii). Hence y = 137, x = 155 is the required least solution in positive integers. The general solution is naturally the same as obtained by the first method.

Example: 4. Find the general solution of the equation 1001 y = 822 x + 1430 in positive integers.[10]

Solution: Here the equation is of the form b y = a x + c, where

a = 822, b = 1001 and c = 1430 and is positive.

Applying the step 2 of the general rule, we get the final two kuṭṭa as follows:

349 (u)

425 (l)

Applying the step 3 of the general rule, we get each of the equal quotients as 607 and remainders as 116 and 143 respectively. Since the quotients are equal, the chain is odd and c is positive, the problem comes under the category 4 (iv). Hence y = 706 and x = 858 is the least solution in positive integers. The general solution is given by x = 1001 m + 858, y = 822 m + 706, m being a whole number.

Alternative Solution: Here a = 822 and c = 1430 have the greatest common divisor $d_1 = 2$. Again b = 1001 and c = 1430 have also the greatest common divisor $d_2 = 143$. Hence the specific rule 3 may be applied to get the solution. For this a = 822 and c = 1430 are divided by $d_1 = 2$. The respective quotients obtained are $a_1 = 411$ and $c_1 = 715$. Again, dividing b = 1001 and $c_1 = 715$ by $d_2 = 143$, we get the quotients as $b_1 = 7$ and $c_2 = 5$ respectively.

Now $a_1 = 411$ and $b_1 = 7$ are divided reciprocally until the remainder comes to unity. Then the application of the step 2 of the general rule rives

58	58	176 (u)
1	3	3 (l)
2	2	

(Initial chain)

Here $$\frac{c\ d_1\ u}{b\ d_1\ u - a\ d_2\ l} = \frac{1430 \times 2 \times 176}{1001 \times 2 \times 176 - 822 \times 143 \times 3} = 1760$$

and $$\frac{c\ d_2\ l}{b\ d_1\ u - a\ d_2\ l} = \frac{1430 \times 143 \times 3}{1001 \times 2 \times 176 - 822 \times 143 \times 3} = 2145$$

Dividing 1760 and 2145 by a = 822 and b = 1001 respectively, we get 116 and 143 as the remainders, each of the equal quotients being 2.

Since the quotients are equal, the chain is odd and c is positive, the problem reduces to category 4 (iv). Hence y = 706, x = 858 is the required solution in positive integers. The general solution is the same as mentioned in the first method.

Example: 5. Find the general solution of the equation 13 y = 5 x + 65 in positive integers.[11]

Solution: Here a = 5, b = 13 and c = 65 and c is positive

Applying step 2 of the general rule, we get

0	0	0	2 (u)
2	2	5	5 (l)
1	2	2	
1	1		

(Initial chain)

The step 3 of the general rule gives 26 and 25 as the quotients and 0 as the remainder in each case. Since the quotients are unequal, the chain is even and c is positive, the problem comes under the category 4 (v). Hence $x = 0$, $y = 5$ is the least solution in positive integers. The general solution is $x = 13m + 0$, $y = 5m + 5$, m being a whole number.

Alternative Solution: Here the quotient obtained, when $c = 65$ is divided by $b = 13$, is 5 and c is positive, hence by the specific rule 4, we get $x = 0$, $y = 5$ as the required least solution[12] in positive integers. The general solution is again the same as mentioned in the first case.

It is pertient to note that Datta and Singh's work[13] contains the basic reasons and principles of rules described above. Dr. A.K. Bag in his research[14] on the ślokas obtained the methods for the solutions of $by = ax \pm 1$ and of $by = ax \pm c$. Knowing the solutions of the reduced equations[15], he also pointed out the methods to get the solutions of the given equation.

However, from all the considerations adduced, it may be concluded that Āryabhaṭa II's general as well as specific rules for the solution of indeterminate equations of the form $by = ax \pm c$; a, b, c being integers, is very usuful in solving the problems in astronomy.

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NOTES AND REFERENCES

1. Shukla, K.S. "Āryabhaṭīya of Āryabhaṭa with the commentary of Bhāskara I and Someśvara" INSA, New Delhi, 1976, pp. 309-339.

2. Datta, B.B. and Singh, A.N., "History of Hindu Mathematics" single volume edition, Bombay, 1962, Part II, pp. 87-161.

3. Sarma, S.R. "The Pūrva gaṇita of Āryabhaṭa's (II) Mahāsiddhānta" Marburg, 1966. Part II, p. 1.

4. Bag, A.K., "Mathematics in ancient and medieval India" Chaukhambha Orientalia, Varanasi, 1979, pp. 207-209.

5. The chain is even or odd according as the number of quotients placed one below the other successively in the initial chain is even or odd.

6. Sources include 'Algebra by Barnard and Child', 'Līlāvatī by L. Jha' and constructed problems satisfying the operations described.

7. The problem similar to Example 6 p. 419 (Barnard and Child's Algebra, New York, 1955) has been considered so that it may fall under category 4 (II).

8. Barnard and Child, "Higher Algebra", MacMillan & Co. Ltd. New York ST. Martin's Press 1955 Ex. 10, p. 419. (The example is considered in modified form so that it may be of the form $b\ y = a\ x \pm c$; a, b, c being integers)

9. Barnard and Child, "Higher Algebra", MacMillan & Co. Ltd. New York ST. Martin's Press 1955, Ex. 7, p. 419.

10. The problem has been constructed to satisfy the fifth operation of Āryabhaṭa II.

11. L. Jha, "Līlāvatī of Bhāskarācārya" Chowkhambha Vidya Bhawan, Varanasi, Fourth edition, 1986, p. 342.

12. In fact, the rule does not hold good when c is negative.

13. Datta, B.B. and Singh, A.N., op. cit. p. 106-109.

14. Bag, A.K., op. cit.

15. The equations obtained, on dividing (a, c) or (b, c) or both by their greatest common divisors in the equation $b\ y = a\ x \pm c$; a, b, c being integers, are called reduced equations.