Indian Journal of History of Science, **29**(4), 1994

SOLUTIONS OF LINEAR ALGEBRAIC EQUATIONS AND SUMS OF FRACTION-ADDITIONS USING SŪTRA METHOD

S. DAS
Bhabha Atomic Research Centre, Theoretical Physics Division
Trombay, Bombay-400 085

(Received 22 July 1994)

Five sūtra algorithms from Bhāratī Kṛṣṇa Tīrthajī's system of Vedic Mathematics are used to solve linear algebraic equations (in one, two, and three variables) and find sums of certain types of fraction-additions. The propriety and the controversy arising from the use of the word Vedic in the title are analyzed and possible reasons for it are given. An etymological analysis of sūtra and the five sūtra formulae is presented. The paper describes the salient features of this novel mathematical system, explains the cryptic meanings of the five sūtra formulae and illustrates their procedures with examples. The examples cover equations with integral, rational, irrational and mixed coefficients. A comparison of the number of mathematical operations and the corresponding calculational time between the sūtra methods and the current theoretical methods shows that for the solution of the same set of problems, the sūtra methods are simpler and faster. Certain advantages of the sūtra methods over the present methods of doing numerical calculations are pointed out. The many applications of the sūtra *parāvartya yojayet* show how the cryptic nature of these sūtras can lend them to many interpretations. Vedic Mathematics as a mathematical system is discussed and its educational potential pointed out. It is suggested that Vedic Mathematics should form part of a course on numerical analysis and the possibility of building a computer using these sūtra algorithms should be explored.

INTRODUCTION

This paper deals with the solution of linear algebraic equations in one, two, and three variables and the determination of sums of fraction-additions using the sūtra method of the late Jagadguru Swāmī Śrī Bhāratī Kṛṣṇa Tīrthajī Mahārāja, former Śaṅkarāchārya of Govardhana Maṭha, Puri[1]. Swāmījī named his new theory and method of calculation Vedic Mathematics or Sixteen Simple Mathematical Formulae from the Vedas for One-line Answers to all Mathematical Problems. The applications of these formulae and procedures to most classical mathematical problems lead to extremely short solutions[2]. According to the author, he discovered these formulae as a result of true realization by actual visualization of certain fundamental mathematical truths, principles and relationships during his eight years of Tapas (concentrated contemplation) in the forests surrounding Sringeri.

Although the present volume is the only work on mathematics that has been left by Swāmījī, he used to say that he had written sixteen volumes, one for each sūtra. He had deposited the manuscripts of the sixteen volumes at the house of one of his disciples from where the manuscripts were lost irretrievably. Subsequently, he re-wrote in 1957 – in one and half months – from his memory the present volume which is a summary of the entire work with an introductory account of the sixteen formulae. Swāmījī had planned to write the other volumes too, but due to the infirmity of old age and poor health, he could not complete the work. The typescript of the present volume was left over by him in U.S.A. in 1958 for publication, and he was to go to U.S.A. for correcting the proofs and supervising the printing personally. But, after his return to India from the foreign tour, his health deteriorated to such an extent that he could not go to U.S.A. Finally, after his expiry in 1960, the typescript was brought back to India from U.S.A. posthumously. The present introductory volume, which is the only work on mathematics that has been bequeathed to posterity by Swāmījī, was first published in 1965 by the Banāras Hindu University, India. All the other works on Vedic Mathematics are lost for good.

The Title "Vedic Mathematics"

The use of the word Vedic in the title of the book is quite misleading[3]. This is because Vedic Mathematics signifies many different types of mathematical activities pertaining to the Vedas[4] and the word Vedic gives the impression that the work belongs to ancient Vedic Mathematical System, and all that the author has done is to acquire them through the usual process of laborious research. But these sūtras are, till now, not found in any of the present recensions of the Atharvaveda. Further, the style of the sūtras, which are in Sanskrit language, indicates that they do not belong to the Vedic period. The contents of the book also indicate that the sūtras are unlikely to be from the Vedas. For instance, topics like differentiation, integration, conjugate hyperbolas and asymptotes were not known in Vedic times. Nonetheless, Swāmījī used to say that the sūtras are contained in the Pariśiṣṭa (the appendix portion) of the Atharvaveda. This inconsistency makes us think that, perhaps, some of the basic materials of Vedic Mathematics are available in the Vedas which Bhāratī Kṛiṣṇa retrieved, synthesized and then composed them in sūtra form in modern Sanskrit using his erudition in Sanskrit language, his knowledge of mathematical terminology gleaned from Sanskritic scientific literature and his scholarship in present-day mathematics. This conjecture of ours is corroborated by Smt. Manjula Devī Trivedī who has stated that the sūtras were, perhaps, reconstructed on the basic of intuitive revelation from materials scattered here and there in the Atharvaveda[5]. In any case, the claimed source and antiquity of the work, as suggested by the title of the book, appear to be untrue and remain a puzzle to the scholars. It is difficult to say as to why Swāmījī chose such a title, but it is possible that he was guided by the following considerations[3] :

(a) Swāmījī wanted to give a sensational title;

(b) He wanted to show his reverential approach towards the Vedas as well as his loyalty to the religious order of which he was one of the Chiefs;

(c) He wanted to show that he accepted the supreme authority of the Vedas;

(d) There, always, has been a tendency in the traditional Hindu orthodoxy of this country to mystify things and give them religious colourings, particularly when it came to creative works. This, perhaps, is done to give the work a stamp of authority and a greater credibility. The Indian literature abounds in legends which give a divine basis to anything creative;

(e) The Vedas or Śrutis are believed to be the words of Brahmā. This mass of knowledge, 'ever the same yet changing ever' was discovered directly based on integral experiences by Āryan personages called Ṛṣis who were seers of thoughts[6]. This knowledge is inaccessible to mankind to-day because we lack a sound understanding of the Vedic scientific techniques and also, because of our incomplete knowledge of Vedic Sanskrit. It is not unlikely that there is a Master Key hidden somewhere – may be in the Vedas themselves – by which it is possible to unravel these secrets. The language barrier has come on the way of our getting at the true meanings of words and we are unable to get proper modern versions of statements which may contain important scientific truths. Even though, real scientific works of those times like the Deva Vidyā and the Bhūta-Vidyā are no longer extant, a large number of statements on important topics are still found scattered in Vedic literature[7,8]. Viewed in this broader perspective and understood in this deeper way, the title Vedic Mathematics may seem justified.

(f) There is also the possibility that a secret version of Atharvaveda exists in the oral form and Swāmījī had knowledge about it.

Whatever it may be, as far as we are concerned, our interest in Vedic Mathematics springs from its scientific content, its intrinsic merit as a delightfully original mathematical system, and also, because of the important place it holds in the twentieth century mathematical history[9].

Main Features of Swāmījī's Vedic Mathematics

Vedic Mathematics forms a self-contained, self-consistent mathematical system and can be considered an alternative formulation of mathematics using the sūtra style. It has the merit of originality and forms a class by itself. The system consists of a set of highly cryptic mathematical propositions coded into 16 sūtras (brief aphorisms) and 13 upa(sub)-sūtras (corollaries). They provide, in condensed form, techniques for solving a large variety of problems in mathematics such as : multiplication, division, decimal fractions, the value of π, the Pythagorean theorem, simple differentiation, etc. The derivations of these sūtras demonstrate great power of insight or intuition on the part of the author, as it has not been possible so far to derive the sūtras and their corollaries, the sub-sūtras by any method of reasoning. A list each of these sūtras and their sub-sūtras is given in Appendix 1 and Appendix

2 respectively.

The system is essentially computational in nature. Its axiomatic approach to numerical computations results in the shortening of mathematical processes a great deal, providing, thereby, fast algorithmic devices capable of doing rapid computations with a speed and an accuracy comparable to or even better than the symbolic method. By avoiding operations with large numbers, the burden of calculations is lighened and the chances of committing numerical errors are reduced. The mathematical processes, however, involve quite a bit of mental calculation[10].

As a computational technique, the sūtra methods of Vedic Mathematics can furnish results upto any degree of accuracy. For example, in finding the complete decimal equivalent of the vulgar fraction 1/49, a computer may fail to give the answer, but the sūtra method gives the correct result in one step[11].

The theorems and the corollaries are highly compact and comprise of just 45 words, 24 for the sūtras and 21 for the sub-sūtras. Couched in Sanskrit language, these words are either single or compound, but mostly one or two compound words. The sūtras are elegant and possess in them the rhyme and beauty of Sanskrit poetry. They are also simple to remember, easy to understand and easy to apply.

The cryptic nature of the sūtras and the sub-sūtras lend them to many interpretations, and the same sūtra can have applications in more than one area of mathematics.

The sūtras provide inherent secrecy, and may, therefore, be suitable for military applications.

Sūtras, Upa-sūtras and their General Principle of applications[12]

The Sanskrit word sūtra (सूत्र) is derived from the root siv meaning to 'sew' and literally means a thread or a string. In figurative sense, it means that which like a thread runs through or holds together everything : a rule, a direction, a short sentence or aphoristic rule. In a broader sense, it means any work or manual which consists of strings of such rules hanging together like threads, or the name of a book. Sūtra is used in the singular to denote both a name for a whole collection of rules, and as a name for a single sūtra. Pāṇini[13,14], the great grammarian always used sūtra in the sense of a whole collection of rules and not as an expression for a single sūtra. The use of the word sūtra as the name for a book might, in all likelihood, have come from the ancient Indian practice of writing on palm-leaves, piercing them and keeping them together by means of a 'string'. This is analogous to the present-day practice of binding a book by putting together pages (leaves). Sūtra form of literature is characterized by great economy of words (high information density) and an artificial, enigmatic form. This minimizes space requirement for storing information and makes learning and transmission of knowledge easy. Sūtra style is unique to Indian literary tradition and has no parallel 'in the entire literature of the world' (Winternitz). The word Upa in Sanskrit is used as a preposition or prefix to verbs and nouns. When prefixed to a noun, it expresses direction towards, nearness, contiguity in space, time, number, degree and relationship but with the idea of subordination and inferiority. In the context of Vedic Mathematics, a sūtra, therefore, means a general rule or formula and an upa(sub)-sūtra means a corollary which provides for special cases or particular types. Vedic Mathematics signifies either the book or simply a collection of rules with each sūtra signifying a truth or theory briefly stated. The power, the simplicity and the quickness of these sūtra formulae lie in the fact that they are statements of general principles which take specific meanings depending upon the situation. We explain this by an example. The ninth sūtra in Vedic Mathematics is : calana-kalanābhyām which literally means 'By differentiation'. This formula is a general statement as such; but its fuller form when applied to the solution of quadratic equations is[15] : Calita-kalita vargo vivecakaḥ which means 'The square (varga) of the differential coefficient (calita-kalita) is (equal to) the discriminant (vivecaka)'. Therefore, according to this rule, the solution of the quadratic equation $ax^2+bx+c=0$ is given by $(2ax+b)^2 = b^2-4ac$ which yields the correct solution. Here, $(2ax+b)$ is the differential coefficient of the left-hand side of the quadratic equation and (b^2-4ac) is its discriminant.

The general principle of applications of the sūtra formulae of Vedic Mathematics is to look for and spot certain characteristics in a given problem, identify the type, and then use the sūtra which is most appropriate. If the problem has certain additional and typical characteristics, it becomes a special case and the upa-sūtra or sub-sūtra is applied instead. The general formula is resorted to only if no special case is involved. Usually, the general formulae are longer than the special ones.

THE ETYMOLOGY OF THE FIVE SŪTRA ALGORITHMS[12]

In the present investigations, the five algorithms to be used are in the form of four sūtras and one sub-sūtra. The four sūtras are :

परावर्त्य योजयेत् (parāvartya yojayet) which cryptically means "Transpose and adjust";

आनुरूप्ये शून्यमन्यत् (ānurūpye śūnyam anyat) which means "If one is in ratio, the other one is zero";

संकलनव्यवकलनाभ्याम् (saṅkalana-vyavakalanābhyām) which means "By addition and by subtraction";

सोपान्त्यद्वयमन्त्यम् (sopāntya-dvayam antyam) whose cryptic meaning is "The ultimate and twice the penultimate".

The one sub-sūtra is :

अन्त्ययोरेव (antyayor eva) whose meaning is "Only the last terms".

The first sūtra parāvartya yojayet consists of one single word (yojayet) and one compound word (parāvartya). The word parāvartya is a blending of two words : parā and vartya. Parā is used as a prefix to nouns and verbs, and indicates 'off, aside or away'. Vartya is derived from the root Vṛt and literally means 'to be turned back or reversed or exchanged'. Hence, parāvartya means 'Having transposed'. The literal meaning of yojayet is 'You should join'. It is a verb form of the word yuj meaning 'to join, to insert, to unite, to connect, to combine, to bring together'. In astronomy, it means 'to come into conjunction with'. In mathematics it can mean add, subtract, multiply, etc. Hence, the sūtra is translated in a general way as : "Transpose and adjust". The second sūtra ānurūpye śūnyam anyat means "If one is in ratio, the other one is zero". The word Ānurūpye is a single one and literally means 'If in proportion'. Śūnyam anyat is a compound word consisting of the two words : Śūnyam, i.e., zero and anyat, i.e., the other one. In the third sūtra saṅkalana-vyavakalanābhyām, the word saṅkalana is derived from the root kal and means 'By addition' and vyavakalana means 'By subtraction'. The ending -bhyām comes from rule of grammar and denotes a dual number. It implies that the twin processes of addition and subtraction should be performed simultaneously. It may be pointed out that in Āryabhaṭīya-Bhāṣya, the word saṁkalanā is used to mean the sum of a series of natural numbers[16]. The fourth sūtra sopāntya dvayam antyam is one compound word consisting of three words : sopāntya meaning 'The penultimate (last but one)', dvayam meaning 'Twice' and antyam meaning 'The ultimate (last one)'. The sub-sūtra antyayoreva is a combination of the two words eva and antyayoh which are blended, by a rule of euphonic combination, to form the compound word antyayor eva. Antyayoḥ is a gen. form of the word antya meaning 'Last in place, in time or in order'. It also denotes the last number of a mathematical series. As an example, the noun antyamūla in arithmetic means the last or greatest root. The word eva is most freqeently used to strengthen an idea expressed by another word, here antyayoḥ. It means 'exactly, same, only, alone, immediately on' and so forth. It implies emphasis, affirmation, etc.

In what follows, we shall start with the simplest of the linear algebraic equations and then progressively go to the more difficult ones.

Parāvartya Yojayet Sūtra and its Aplications

One of the most versatile of the sūtras in the system of Vedic Mathematics is the Sūtra parāvartya yojayet. Its underlying principle of application, when applied to the solution of algebraic equations, is : gather all the terms involving the single real variable (say, x) to the left side of the equation and all the independent terms to the right side of the equation with the condition that every transposition during the process produces a sign inversion, i.e., '+' becomes '–' and vice versa; 'x' becomes '÷' and vice versa. This reduces the number of transposition operations.

Equations of the form $ax+b = cx+d$

This is the commonest type of equation which is solved by transposition. It follows from what has been said above that

$$x = (d-b)/(a-c) \tag{1}$$

Problem 1

$2x+7 = 13x-10$. Therefore, $x = (-10-7)/(2-13) = -17/-11 = 17/11$

Equations of the form $(ax+b)/(cx+d) = p/q$

$$x = (pd-bq)/(aq-cp) \tag{2}$$

Under the restrictive condition that $p=q$, eq. (2) reduces to eq. (1).

Problem 2

$(21x+2)/(5x+9) = 6/7$. Therefore, $x = (54-14)/(147-30) = 40/117$

Equations of the form $(x+a)(x+b) = (x+c)(x+d)$

This is another common type of equation in which both sides of the eqation contain two binomial factors of degree one each. Using the parāvartya sūtra, we immediately write

$$x = (cd-ab)/(a+b-c-d) \tag{3}$$

$= 0$ if $cd=ab$, i.e., the product of the absolute terms are the same on both sides.

Problem 3

$(\sqrt{19}x+1)(\sqrt{19}x+11) = (\sqrt{19}x+13)(\sqrt{19}x+6)$

Therefore, $\sqrt{19}x = (78-11)/(12-19) = -67/7 \Rightarrow x = -67/7\sqrt{19}$

Equations of the form $m/(x+a) + n/(x+b) = 0$

Application of the parāvartya sūtra gives

$$x = -(bm+an)/(m+n) \tag{4}$$

Problem 4

$2/[(13x/3)-1] + 1/[(13x/3)+3] = 0$

Therefore, $13x/3 = -(6-1)/(2+1) = -5/3 \Rightarrow -5/13$

The above procedure can be extended to equations having any number of terms provided a suitable condition is imposed so that the order of the equation does not exceed unity. For example, in the equation

$m/(x+a) + n/(x+b) + p/(x+c) = 0,$

which has three terms, we can write

$x = -(mbc + nca + pab) / [m(b+c) + n(a+c) + p(a+b)],$

provided that $m+n+p = 0$. Otherwise, the equation becomes a quadratic in x.

Merger type of simple equations

These refer to equations of the type

$\Sigma_i \, p_i/(x+a_i) = r/(x+b)$

Single step merger

The method is applicable to equations in which the sum of the numerators on the LHS equals the single numerator on the RHS, i.e., $\Sigma_i \, p_i = r$. The procedure consists in merging the fraction on the RHS into the LHS and equating the RHS to zero till the equation contains only two terms. This derived equation is solved for x. Merging is accomplished by subtracting the independent term (b) of the binomial on the RHS from the absolute terms (a_i 's) of the binomials on the left, and then multiplying these results by their corresponding numerators (p_i 's) successively. The results so obtained become the new numerators of the fractions on the LHS, the denominators remaining unchanged. The simplest among these types of equations is where there are only two terms on the LHS, i.e., the equation is of the form

$p_1/(x+a_1) + p_2/(x+a_2) = r/(x+b)$

Problem 5

$5/(x-2) + 7/(x+4) = 12/(x-9)$

Since, $p_1+p_2 = 12 = r$, the sūtra applies and the derived equation is

$5[-2-(-9)]/(x-2) + 7[4-(-9)]/(x+4) = 0$

or, $35/(x-2) + 91/(x+4) = 0$ Or, $5/(x-2) + 13/(x+4) = 0$

Using eq. (4), we get

$x = -[(4)\ (5) + (-2)\ (13)]/(5+13) = -(20-26)/18 = 6/18 = 1/3$

Disguises

These refer to equations which are not readily seen to be of the single step merger type. They are of two kinds : (i) Co-efficients of x equal; and (ii) Co-efficients of x unequal. To solve such equations, first the applicability of the parāvartya sūtra is tested. This is done by equalising the x-coefficients in the denominator – if necessary – using the LCM method, and then dividing the numerator of each term by the corresponding x-coefficient in the denominator. If the sum is the same on both sides of the equation, the sūtra applies. The equation is then written in its proper form and we proceed as before. We demonstrate the procedure by solving an equation in which the co-effieients of x are unequal. The equation where x-coefficients are equal is a special case of this.

Problem 6

$2/(\sqrt{3}x-1) + 5/(\sqrt{2}x+7) = (2\sqrt{2}+5\sqrt{3})/(\sqrt{6}x+9)$

Equalising the x-coefficients in the denominator,

$2\sqrt{2}/(\sqrt{6}x-\sqrt{2}) + 5\sqrt{3}/(\sqrt{6}x+7\sqrt{3}) = (2\sqrt{2}+5\sqrt{3})/(\sqrt{6}x+9)$

LHS $= 2\sqrt{2}/\sqrt{6} + 5\sqrt{3}/\sqrt{6} = (2\sqrt{2}+5\sqrt{3})/\sqrt{6}$ and RHS $= (2\sqrt{2}+5\sqrt{3})/\sqrt{6}$

Equality is satisfied. Therefore, applying the sūtra, we get

$2\sqrt{2}(-\sqrt{2}-9)/(\sqrt{6}x-\sqrt{2}) + 5\sqrt{3}(7\sqrt{3}-9)/(\sqrt{6}x+7\sqrt{3}) = 0$

Using eq. (4), we get

$\sqrt{6}x = -[14\sqrt{6}(-\sqrt{2}-9)-5\sqrt{6}(7\sqrt{3}-9)]/[2\sqrt{2}(-\sqrt{2}-9)+5\sqrt{3}(7\sqrt{3}-9)]$

$= -\sqrt{6}(-14\sqrt{2}-126-35\sqrt{3}+45)/(-4-18\sqrt{2}+105-45\sqrt{3})$

$= -\sqrt{6}(-81-14\sqrt{2}-35\sqrt{3})/(101-18\sqrt{2}-45\sqrt{3})$

$\therefore x = (81+14\sqrt{2}+35\sqrt{3})/(101-18\sqrt{2}-45\sqrt{3})$

Multiple step merger

The single step merger method as described above can be extended to equations having more than two terms on the LHS. The procedure is a repeated application of the method of single step merger.

Problem 7

$$4/(2x-1) + 27/(3x-1) + 125/(5x-1) = 144/(4x-1)$$

Equalising the x-coefficients in the denominator, and then dividing throughout by 60, we get

$$2/(60x-30) + 9/(60x-20) + 25/(60x-12) = 36/(60x-15)$$

$$\text{LHS} = 2/60 + 9/60 + 25/60 = 36/60 = \text{RHS}$$

By merger, the derived equation is

$$-30/(60x-30) + [-45/(60x-20)] + 75/(60x-12) = 0$$

$$\text{Or, } 2/(60x-30) + 3/(60x-20) = 5/(60x-12)$$

LHS = 2/60 + 3/60 = 5/60 = RHS. Therefore, again by merger

$$-36/(60x-30) + [-24/(60x-20)] = 0$$

$$\text{Or, } 3/(60x-30) + 2/(60x-20) = 0$$

Using eq. (4), we get

$$60x = -[(-20)3-(30)2]/(3+2) = -(-120/5) \Rightarrow x = 2/5$$

The above merger method can be generalized to solve merger type of simple equations having any number of terms on the LHS. The general merger formula is

$$[p_1(a_1-b) \ldots\ldots (a_1-a_5)(a_1-a_4)(a_1-a_3)]/(x+a_1)$$
$$+[p_2(a_2-b) \ldots\ldots (a_2-a_5)(a_2-a_4)(a_2-a_3)]/(x+a_2) = 0$$

and the general solution is

$$x = -[a_2p_1(a_1-b) \ldots\ldots (a_1-a_5)(a_1-a_4)(a_1-a_3)$$
$$+a_1p_2(a_2-b) \ldots\ldots (a_2-a_5)(a_2-a_4)(a_2-a_3)]/$$
$$[p_1(a_1-b) \ldots\ldots (a_1-a_5)(a_1-a_4)(a_1-a_3)$$
$$+p_2(a_2-b) \ldots\ldots (a_2-a_5)(a_2-a_4)(a_2-a_3)]$$

Applying these formulae to Problem 5, the final derived equation is

$$5(-2+9)/(x-2) + 7(4+9)/(x+4) = 0 \quad \text{Or,} \quad 5/(x-2) + 13/(x+4) = 0$$

and the solution is

$$\begin{aligned} x &= -[4(5)\ (-2+9) + (-2)\ (7)\ (4+9)]/[5(-2+9) + 7(4+9)] \\ &= -(20.7-2.7.13)/(7.5+7.13) = -(20-26)/(5+13) = -(-6)/18 = 1/3 \end{aligned}$$

Complex merger

We now discuss equations which contain two terms on each side and the coefficients of x are not all equal. The procedure is as follows :

Step 1 Check if $\Sigma_i(N_i/C_i)$ is the same on both sides of the equation. Here, N_i denotes the numerator and C_i the x-coefficient in the denominator of the ith fraction.

Step 2 If the above condition is satisfied, equalise the x-coefficient by the LCM method to obtain the derived equation.

Step 3 Arrange the equation in such a way that the numerator on both sides of the final derived equation is the same.

Step 4 If we pass the above three tests, Parāvartya sūtra becomes applicable and we write,

$$\underset{i=1}{\overset{2}{\pi}} D_i \text{ on LHS} = \underset{i=1}{\overset{2}{\pi}} D_i \text{ on RHS } (D_i = \text{Denominator of the ith fraction})$$

Step 5 Solution of the equality in Step 4 furnishes the value for x.

Problem 8

$$51/(3x+5) - 68/(4x+11) = 52/(4x-15) - 39/(3x-7)$$

We note that

$$\Sigma_i N_i/C_i \text{ on LHS} = 51/3 - 68/4 = 17 - 17 = 0$$

$$\Sigma_i N_i/C_i \text{ on RHS} = 52/4 - 39/3 = 13 - 13 = 0$$

Therefore, we equalise the x-coefficients and obtain the derived equation as

$$204/(12x+20) - 204/(12x+33) = 156/(12x-45) - 156/(12x-28)$$

N on LHS = 204 x 13 = 12 x 17 x 13

N on RHS = 156 x 17 = 12 x 17 x 13

Therefore, the above equation is the final derived equation and the parāvartya sūtra is applicable. We write

$$(12x+20)\ (12x+33) = (12x-45)\ (12x-28)$$

Using eq. (3), we obtain

$$12x = (45x28 - 20x33)/(53+73) = 600/126 \quad \Rightarrow \quad x = 25/63$$

Simultaneous simple equations

For the solution of simultaneous equations of the type

$$ax + by = c\}$$
$$px + qy = r\},$$

methods that are currently in use are the cross-multiplication method and the substitution or elimination method. Of the two, the former is less time consuming than the latter. But, it sufferes from the drawback arising due to a confusion in the sign convention. Consequently, for the majority of the problems, the elimination method is followed.

We shall, here, discuss for the solution of simple simultaneous equations of all types a general rule from the system of Vedic Mathematics. The rule is an application of the parāvartya sūtra and is a cyclic one. Its advantages are : it does not suffer from the short-comings of the current methods as mentioned above, and entails less time and labour.

(i) To determine the value of x, cross-multiply the y-coefficients and the independent terms first forward, i.e., rightward and then backward, i.e., leftward. This gives the numerator. The denominator is obtained by going from the y-coefficient in the upper row across to the x-coefficient in the lower row first backward and then forward.

$$\therefore\ x = (br-cq)/(bp-aq)$$

(ii) To determine the value of y, follow the cyclic system, i.e., start with the independent term on the upper row towards the x-coefficient on the lower row and then the x-coefficient in the upper row towards the independent term on the lower row to get the numerator. The denominator is invariably the same as for x.

$\therefore\ y = (cp-ar)/(bp-aq)$

In all the operations, the connecting link between the two cross-products is always the '–' sign.

Problem 9

$$\left.\begin{aligned}\sqrt{19}x + (5/2)y &= 7\\ 3x + \sqrt{13}y &= 4/9\end{aligned}\right\}$$

Using the above rule, we obtain

$$\begin{aligned}x &= [(5/2)(4/9) - 7\sqrt{13}]/[(5/2)3 - \sqrt{19}\sqrt{13}]\\ &= [(10/9) - 7\sqrt{13}]/[(15/2)-\sqrt{247}] = (2/9)(10-63\sqrt{13})/(15-2\sqrt{247})\\ y &= (21 - 4\sqrt{19}/9)/(15/2 - \sqrt{247}) = (2/9)(189 - 4\sqrt{19})/(15-2\sqrt{247})\end{aligned}$$

Fractional equations of the type

$$[1/f_1(x)f_2(x)] + [1/f_2(x)f_3(x)] + [1/f_3(x)f_1(x)] = 0$$

where, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are the binomial factors of a particular cyclical kind.

Using the Parāvartya sūtra, the solution of the above equation is

$$x = \sum_{i=1}^{m} N_i I_i \Big/ \sum_{i=1}^{m} N_i C_i$$

where, N_i denotes the numerator of the ith fraction; I_i, C_i are the independent term with the sign reversed and the x-coefficient respectively of the absent binomial factor in the denominator of the ith fraction. The rule can be applied to equations having any number of factors in the denominator provided that the denominator is expressed as a product of binomials which are in cyclic order.

Problem 10

$$[1/(x-3)(x-4)] + [3/(x-4)(x-9)] + [5/(x-9)(x-3)] = 0$$

Using the above rule, we get

$$x = [1(-(-9)+3\{-(-3)\}+5\{-(-4)\}]/[1(1)+3(1)+5(1)] = (9+9+20)/(1+3+5) = 38/9$$

If in the original equation, RHS $\neq 0$, adjustments are made to make RHS $= 0$ as in the problem below.

Problem 11

$$(8x+10)/(2x+1)(4x+3)+(24x+38)/(4x+3)(6x+5)+(12x+16)/(6x+5)(2x+1) = 3/x$$

Multiply throughout by x, transpose 3 to the left and rearrange. We obtain

$$[\{(8x^2+10x)/(2x+1)(4x+3)\}-1]+[\{(24x^2+38x)/(4x+3)(6x+5)\}-1]$$
$$+[\{(12x^2+16x)/(6x+5)(2x+1)\}-1] = 0$$

Or, $-3/(2x+1)(4x+3) - 15/(4x+3)(6x+5) - 5/(6x+5)(2x+1) = 0$

$$\therefore x = [(-3)(-5)+(-15)(-1)+(-5)(-3)]/(-3x6-15x2-5x4)$$
$$= (15+15+15)/(-18-30-20) = -45/68$$

ĀNURŪPYE ŚUNYAM ANYAT SŪTRA AND ITS APPLICATIONS

If the simultaneous equations are such that the ratio of the coefficients of one of the unknown quantities is equal to the ratio of the independent terms, the above sūtra tells us to set the second unknown quantity equal to zero. Consequently, the term containing the first unknown quantity is equated to the absolute term on the right.

Problem 12

$$(5/11)x + (\sqrt{7}/4)y = 15\}$$
$$19x + \sqrt{7}\ y = 60\}$$

Here, $(\sqrt{7}/4)/\sqrt{7} = 1/4$ and $15/60 = 1/4$. Hence, by the above rule

$$x = 0$$

Therefore, $\sqrt{7}y = 60 \Rightarrow y = 60/\sqrt{7}$

Solution of the above problem clearly demonstrates that the sūtra method avoids multiplication of numbers. The procedure should, therefore, be particularly useful while dealing with simultaneous simple equations involving large numbers as the problem below shows.

Problem 13

$$499x + 172y = 212\}$$
$$9779x + 387y = 477\}$$

It is noted that

$212/477 = (4x53)/(9x53) = 4/9$
$172/387 = (4x43)/(9x43) = 4/9$

Hence, it follows that $x = 0$

Therefore, $172y = 212 \Rightarrow y = (4x53)/(4x43) = 53/43$

The above procedure can be extended to equations having any number of unknown quantities. Consider the following set of three equations in three variables.

Problem 14

$$
\begin{aligned}
97\ x + ay + 43\ z &= am\} \\
49979\ x + by + (p+q)\ z &= bm\} \\
49(a-d)^3\ x + cy + (m-n)^3\ z &= cm\}
\end{aligned}
$$

In the first two equations, $ay/by=a/b$ and $am/bm=a/b$

In the last two equations, $by/cy=b/c$ and $bm/cm=b/c$

$x = 0$ and $z = 0$. Consequently, $ay = am \Rightarrow y = m$

THE SAṄKALANA-VYAVAKALANĀBHYĀM SŪTRA

If the simultaneous linear equations are such that the x and the y-coefficients are found interchanged, their solutions can easily be arrived at by the application of the above sūtra. The method consists in adding and subtracting the equations repeatedly.

Problem 15

$$
\begin{aligned}
(23/2)x - \sqrt{41}y &= 105\} \\
\sqrt{41}x - (23/2)y &= 99\}
\end{aligned}
$$

Addition gives, $[(23/2) + \sqrt{41}]\ [x-y] = 204$ Or, $x-y = 408/(23+2\sqrt{41})$

Subtraction gives, $[(23/2) - \sqrt{41}]\ [x+y] = 6$ Or, $x+y = 12/(23-2\sqrt{41})$

Repetition of the above process gives

$$
\begin{aligned}
2x &= 408/(23+2\sqrt{41}) + 12/(23-2\sqrt{41}) \\
&= (408x23+12x23-816\sqrt{41}+24\sqrt{41})/(23x23-4x41) \\
&= (23x420-792\sqrt{41})/(529-164) \Rightarrow x = (4830-396\sqrt{41})/365
\end{aligned}
$$

Substituting for x in the second equation, we get

$$-(23/2)y = 99-\sqrt{41}x = 99-\sqrt{41}(4830-396\sqrt{41})/365$$

$$= (36135-4830\sqrt{41}+16236)/365 = (52371-4830\sqrt{41})/365$$

Therefore, $y = 2(4830\sqrt{41}-52371)/(23x365) = (420\sqrt{41}-4554)/365$

We thus see that even if the coefficients are fairly large and complex, elaborate multiplication, subtraction and division are avoided.

THE SOPĀNTYA-DVAYAM ANTYAM SŪTRA

This sūtra is used to solve equations of the type

$$1/Q(x)Q_1(x) + 1/Q(x)Q_2(x) = 1/Q(x)Q_3(x) + 1/Q_1(x)Q_2(x) \qquad (5)$$

where, $Q(x)$, $Q_1(x)$, $Q_2(x)$, $Q_3(x)$ are polynomial factors of degree one and are in arithmetic progression. The sūtra tells us to obtain the value of x from the equation

$$L+2P = 0$$

where, L is the ultimate or the last polynomial factor, i.e., $Q_3(x)$ and P is the penultimate or the last but one factor which is $Q_2(x)$.

Problem 16

$$1/(3x^2+5x+2) + 1/(5x^2+8x+3) = 1/(7x^2+11x+4) + 1/(15x^2+19x+6)$$

Or, $1/(x+1)(3x+2) + 1/(x+1)(5x+3) = 1/(x+1)(7x+4) + 1/(3x+2)(5x+3)$

Here, $Q(x) = x+1$, $Q_1(x) = 3x+2$, $Q_2(x) = 5x+3$ and $Q_3(x) = 7x+4$

are all polynomial factors of degree one and are in arithmetic progression with a common difference of 2x+1. Therefore, all the conditions are satisfied and the sūtra applies. We write

$(7x+4)+2(5x+3)\ 0$ Or, $x=(-6-4)/(7+10)=-10/17$ [By eq. (1)]

In this problem, the equation is actually a quadratic. It has been specifically chosen to show that the formula sopāntya-dvayam antyam is applicable to both linear and quadratic equations. In the latter case, however, the sūtra gives only one solution correctly (x=−10/17 in the present case). The other solution is x=−1/2 (see p. 44, Annexure 3). Applicability of the formula to linear equations is obvious.

ANTYAYOREVA SUB-SŪTRA AND ITS APPLICATIONS

We now discuss various types of miscellaneous linear equations which are amenable to solution by the sūtra formula antyayoreva.

EQUATIONS OF THE TYPE IN WHICH THE RATIO OF THE NUMERATOR TO THE DENOMINATOR ON THE LHS (EXCLUDING THE INDEPENDENT TERMS) EQUALS THE RATIO OF THE ENTIRE NUMERATOR TO THE ENTIRE DENOMINATOR ON THE RHS

In such situations, the sūtra tells to equate the absolute terms on the LHS to the quantity on the RHS and then solve the equation.

Problem 17

$$[(x/2)+\sqrt{5}]^2/[(x/3)+\sqrt{7}]^2 = (9x+36\sqrt{5})/(4x+24\sqrt{7})$$

Ratio on the LHS (excluding the independent terms)

$$= [(x^2/4)+\sqrt{5}x]/[(x^2/9)+(2\sqrt{7}/3)x]$$

$$= (x+4\sqrt{5})9/4(x+6\sqrt{7}) = (9x+36\sqrt{5})/(4x+24\sqrt{7}) = \text{RHS}$$

Therefore, the rule applies and we write

$$5/7 = (9x+36\sqrt{5})/(4x+24\sqrt{7})$$

Using eq. (2), we get

$$x = (5\text{x}24\sqrt{7}-7\text{x}36\sqrt{5})/(9\text{x}7-4\text{x}5) = (120\sqrt{7}-252\sqrt{5})/43$$

Fraction-additions of the type where the factors in the denominators are either in Arithmetic Progression or are related to one another in a special manner as in Summation of Series.

The factors of the denominators in the fractions can either only be the independent terms or only the x-coefficients or may consist of both. Sums of such series are readily obtainable by the Antyayoreva sūtra which is here interpreted as follows. The sum of this series is a fraction whose numerator is the sum of the numerators in the series, and denominator is the product of the two ends, i.e., the first and the last binomials. The rule applies indefinitely to as many terms and to as many varieties as one may please.

Problem 18

$$1/(x^2+x+1)(x^2+2x+2) + 1/(x^2+2x+2)(x^2+3x+3) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The sūtra applies and the sum of the first four terms is

$$S_4 = (1+1+1+1)/(x^2+x+1)(x^2+5x+5) = 4/(x^2+x+1)(x^2+5x+5)$$

In this problem, the denominators are quadratics and the progression is with regard to both the x-coefficient and the absolute term.

Problem 19[17]

$$1/(1+x)(1+2x) + 1/(1+2x)(1+3x) + 1/(1+3x)(1+4x) + \ldots\ldots$$

It is easily seen that the sūtra applies and the nth term being equal to $1/[1+nx][1+(n+1)x]$, we write the sum to n terms (S_n) of the above series as

$$S_n = (1+1+1+\ldots\text{ to n terms})/[1+x][1+(n+1)x] = n/[1+x][1+(n+1)x]$$

Problem 20[17]

$$1/(1+x)(1+ax) + a/(1+ax)(1+a^2x) + a^2/(1+a^2x)(1+a^3x) + \ldots\ldots$$

The nth term of the infinite series = $a^{n-1}/(1+a^{n-1}x)(1+a^nx)$. The sūtra applies and we write the sum to n terms (S_n) of the infinite series as :

$$S_n = (1+a+a^2+\ldots+a^{n-1})/(1+x)(1+a^nx) = (1-a^n)/(1-a)(1+x)(1+a^nx)$$

Problem 21

$$1/56 + 1/72 + 1/90 + 1/110 + \ldots\ldots$$

$$= 1/(7\text{x}8) + 1/(8\text{x}9) + 1/(9\text{x}10) + 1/(10\text{x}11) + \ldots\ldots$$

The sūtra applies and we write the sum of the first four fractions as

$$S_4 = (1+1+1+1)/(7\text{x}11) = 4/77$$

Equations of the type where each numerator is the difference of the two binomial factors in its denominator.

This type of fraction-additions are met with in Summation of Series and can be tackled with the same Antyayoreva formula. The sum of the series is a fraction whose numerator is the difference of the two ends, i.e., the first and the last binomial factors in the denominator, and the denominator is the product of the same two ends.

Problem 22

$$3/70 + 9/190 + 27/874 + 99/6670 + \dots\dots$$

This is rewritten as

$$= -(7-10)/(7\mathrm{x}10) - (10-19)/(10\mathrm{x}19) - (19-46)/(19\mathrm{x}46)$$

$$-(46-145)/(46\mathrm{x}145) - \dots\dots$$

Applying the antyayoreva sūtra, we write the first (7) and the last (145) terms to obtain the sum of the first four fractions as :

$$S_4 = -(7-145)/(7\mathrm{x}145) = 138/1015$$

The following two points may be noted in this problem :

(i) If the last term were 7 instead of 145, we would get the sum

$$S_4 = -(7-7)/(7\mathrm{x}7) = 0$$

(ii) As in the LHS, the RHS (138/1015) also has the characteristic property that the numerator (138) is the difference of the two binomial factors (7 and 145) in the denominator.

Problem 23

$$(a-b)/(px+a)(px+b) + (b-c)/(px+b)(px+c) + (c-d)/(px+c)(px+d) + \dots\dots$$

It is seen that the numerator in each fraction is the difference of the two binomial factors in the denominator. Therefore, we write

$$S_3 = [(px+a)-(px+d)]/(px+a)(px+d) = (a-d)/(px+a)(px+d)$$

Solutions of the problems above clearly show that the sūtra method avoids the tiresome procedure of finding the LCM, the multiplications, the divisions, the additions, the cancellations, etc.

Discussions and Conclusions

The methods of solving linear algebraic equations and of finding sums of certain types of fraction-additions using the sūtra algorithms of Vedic Mathematics have been discussed and the unique features of the system described. We have seen how the simple techniques of Vedic Mathematics for doing algebraic computations give quick results. We now discuss Vedic Mathematics as a mathematical system.

Table 1 : Comparisons of The Number of Operations and The Calculational Time Between The Current Methods and The Sūtra Methods (SM = Sūtra Method, CM = Current Method, EM = Elimination Method, GR = General Rule, PF = Partial Fraction Method, LCM = LCM Method)

Reference problem number	Calculational time (approximate) SM (t_s)	Calculational time (approximate) CM (t_c)	t_c/t_s	CT	Number of operations SM (N_s)	Number of operations CM (N_c)	N_c/N_s
1	5 s	9 s	1.8	9 s	3	5	1.7
2	18 s	22 s	1.2	22 s	9	12	1.3
3	15 s	30 s	2.0	70 s	9	16	1.8
4	20 s	50 s	2.5	70 s	6	15	2.5
5	30 s	80 s	2.7	80 s	10	35	3.5
6	180 s	250 s	1.4	330 s	30	42	1.4
7	210 s	230 s	1.1	360 s	57	125	2.2
8	210 s	220 s	1.1	240 s	61	65	1.1
9	100 s	240 s (EM)	2.4	290 s	17	25	1.5
10	20 s	35 s	1.7	60 s	14	16	1.1
11	150 s	200 s	1.3	360 s	38	67	1.8
12	15 s	40 s (EM)	2.7	150 s	5	9	1.8
	180 s (GR)	40 s (EM)	0.22	150 s	48 (GR)	9	0.2
	40 s (GR)*	40 s (EM)	1.0	150 s	5 (GR)	9	1.8
13	45 s	260 s (EM)	5.8	1200 s	15	86	5.7
	80 s (GR)	260 s (EM)	3.2	1200 s	34 (GR)	86	2.5
14	30 s	500 s	16.6	900 s	5	25	5.0
15	240 s	420 s (EM)	1.2	420 s	70	85	1.2
	180 s (GR)	420 s (EM)	2.3	420 s	62 (GR)	85	1.3
16	110 s	285 s	2.6	350 s	30	82	2.7
17	100 s	480 s	4.8	330 s	23	91	3.9
18	10 s	140 s	14.0	420 s	3	27	9.0
19	10 s	90 s	9.0	145 s	3	19	6.3
20	15 s	310 s	20.7	480 s	3	41	13.7
21	15 s	200 s (LCM)	13.3	300 s	8	42	5.2
	15 s	45 s (PF)	3.0		8	23	2.9
22	45 s	650 s (LCM)	14.4	1800 s	25	187	7.5
	45 s	70 s (PF)	1.5		25	43	1.7
23	40 s	600 s (LCM)	15.0	900 s	7	75	10.7
	40 s	85 s (PF)	2.1		7	19	2.7

The first of these is its utilitarian aspect as a fast computational tool. In Table 1, we present comparisons of the number of operations performed and the corresponding calculational time between the current methods and the sūtra methods for the problems dealt with in this paper. The operations included in the counting were : cancellations and collections of like terms, transpositions, extraction of common factors, and finding the LCM in addition to the fundamental mathematical operations. The calculational time given are the real time, i.e., the clock time. In all, 23 problems were solved. In problems 3 and 6, the co-efficients in the equation are irrational numbers, in problem 4 the co-efficients are rational numbers, in problem 15 the co-efficients are both rational and irrational, in problems 9, 12, 17 the co-efficients are of mixed type, and problems 14, 20, 23 are of general types of equations. In rest of the problems, the co-efficients are integers.

It is seen from Table 1 that the ration (t_c/t_s) of the two computational times ranges mostly from about 1.2 to about 15. In four problems (18, 21, 22, 23) using the Antyayoreva formula, this ratio is around 14 and in problem 20 it is as much as 20. In one application (problem 14) of the ānurūpye śūnyam anyat sūtra, $t_c/t_s \sim 17$. These results demonstrate that the sūtra algorithms are much faster than the current methods. Let us now compare this time-ratio with the corresponding ratio (N_c/N_s) of the number of operations. We observe that, in the majority of the cases, the reductions in computational time by sūtra methods follow closely the corresponding reductions in the number of operations. However, in a few cases (Problems 14, 18, 20 to 23), the pattern is different. Here, the gain in calculational speed, as one goes from current methods to the sūtra methods, is much more than the corresponding savings in the number of operations would suggest. For instance, in problem 14, N_s is only one-fifth of N_c, but the sūtra method is about 17 times faster than the current method. This highlights the shortness and the simplicity of the processes involved in the sūtra methods. In problem 10, difference in the number of operations between the two methods is marginal, but not so the difference in the calculational time. In problem 12, the number of operations, when the general rule is used, is more in sūtra method than in the current method as a result of which the sūtra method is considerably slower. This was because the general sūtra formulae for determining x and y were used as such. But, when the value of x=0 obtained using the general sūtra formula was substituted in one of the simultaneous equations ($19x+\sqrt{7}y=60$) to determine y instead of calculating it from the general sūtra formula, the sūtra method was found to be as fast (40 sec) as the current method. This is shown by an asterisk mark in the third row under problem 12. The large values of t_c obtained in problems 14 to 17, 20, 22 (LCM) and 23 (LCM) are either due to the current methods being much longer or because larger-digit numbers are involved in the calculation or both. It is interesting to note that in problem 15 for the solution of simultaneous equations, the general formula takes less time (180 sec) than the special formula (240 sec). These results and observations show that Vedic Mathematics is usually much better when working with special types of equations than with the general ones. This means that one should be discrete in using the general formulae and use them only when the equations do not possess the characteristic(s) that make one of the special formulae applicable. The values of t_c/t_s quoted in Table 1 are on the lower side and are likely to increase owing to the following reasons. First, our experience of working with Vedic Mathematics is very little at the moment. With greater use, t_s will further decrease. Second, in Vedic Mathematics, mental calculations play quite a significant role in the mathematical processes. Consequently, many of the mathematical operations and steps that the author wrote can be skipped. This would considerably decrease t_s. Third, t_c, on an average, will increase when the problems are worked out by students. This is shown in the fifth column of Table 1 under the heading CT. They were worked out by Varsha[18] using the conventional methods (See Appendix 3). It is seen that the increase in CT over t_c is by a factor ranging from 1 to 5. These results are sufficiently encouraging to prompt one explore the possibility of developing soft-ware based on these sūtra algorithms for use in high speed computers[19-21] or in Personal Computers[22]. The possibility of modifying existing programs written for the solution of problems of applied nature or popular commercial software such as the Mathematica[23,24] can also be explored. In an earlier paper[11], we demonstrated the possibility of fruitfully incorporating some of these sūtra algorithms into modern computers. For this, the square root of the number 119716 was evaluated using both the sūtra method and the Library Function SQRT(X). It was found that even if the sūtra algorithm was programmed in Fortran Language, the average execution time was same in both the cases, namely, 31 milliseconds. The process of developing soft-ware using the sūtra algorithms may spark off clue(s) to the development of a new programming language. Finally, efforts may be made to build a computer using the sūtra algorithms of Vedic Mathematics. It is important to point out that speed may not always be the most important criterion in making a choice of an algorithm. The ease with which a calculation can be performed may be of greater importance in many situations. In such cases, Vedic Mathematics can become very handy by lightening the burden of computations.

The second aspect of Vedic Mathematics is the mathematical one. Swāmī Bhāratī Kṛṣṇa has said that the sūtras apply to all mathematical problems[1]. But, since only one volume on the subject has survived, there is incentive to delve deeper into the subject and see how the areas of applicability of the sūtra formulae can be widened, and more results are derived using the existing formulae or more of similar results are developed[25,26]. The present investigation is a modest attempt in this direction. This study can be further extended to the field of complex numbers and a comparison of their computational efficiencies with the current numerical techniques can be made. Extension or modification of the relevant sūtra algorithm(s) to find sums of mathematically divergent series, which occur in the computation of Dirac's atomic hydrogen wave functions of the continuum, may also prove to be interesting. The late Prof. Bhatnagar[27] gave proofs of many of the formulae given by Swāmījī. Recently, Bhanu Murthy[28] has given proofs of some results. But as yet no proofs of the 16 sūtras exist. There is, therefore, the need for a comprehensive work giving proofs of all the sūtras wherein all the results in Vedic Mathematics can be written.

The third and the most original aspect of Śaṅkarāchārya's Vedic Mathematics is its sūtra style of formulation. These 16 sūtras and 13 upa-sūtras were constructed and composed by the author himself in modern Sanskrit but employing the old sūtra-style[3,4,12,29–31]. While the cryptical quality of Vedic Mathematics comes from the basic nature of sūtra style of composition, the applicability of the very same sūtra to more than one area of mathematics depends on how the sūtra is interpreted. Consequently, the real problem of diversification of the area of applications of Vedic Mathematics is how to bring the various branches of mathematics within the orbit of the original sūtra style, since this requires extraordinary power of intellection and intuition quite apart from excellent scholarship in Mathematics and profundity in Sanskrit language, particularly in grammar. It is worthwhile pointing out that the more than one meaning that a sūtra can have is not something that is unique to Vedic Mathematics. In Sanskrit literature, it is not unusual for words and expressions to bear more than one meaning each as good as the other. For instance, the word Upaniṣad[32] has four meanings, each as appropriate as the other. The compound word Abhijñānaśakuntalam (अभिज्ञानशकुन्तलम्), the title of a play by Kalidāsa can be interpreted in four different ways, each of which is equally significant[33]. Nevertheless, Vedic Mathematics appears to have an empirical base when one notes that Bhāratī Kṛṣṇa seems to have suppressed the proofs of many mathematical formulae and then abstracted them into the form of sūtras. But, in this process, he has gained in two respects : greater computational economy (by eliminating the intermediate mathematical steps), and simplification of learning and the oral transfer of information. As example, consider the problem 5 in the text :

$$5/(x-2) + 7/(x+4) = 12/(x-9)$$

It can be rewritten as

$$5/(x-2) + 7/(x+4) = 5/(x-9) + 7/(x-9)$$

By transposition, we get

$$5/(x-2) - 5/(x-9) + 7/(x+4) - 7/(x-9) = 0$$

Or, $(-7x5)/(x-2)(x-9) + 7(-13)/(x+4)(x-9) = 0$

Cancelling out the common factor $-7/(x-9)$ throughout, we get

$$5/(x-2) + 13/(x+4) = 0$$

which is exactly the derived equation we got by the merger method.

As another example, consider algebraic equations of the type in eq. (5) :

$$1/(Q.Q_1) + 1/(Q.Q_2) = 1/(Q.Q_3) + 1/(Q_1.Q_2)$$

This can be rewritten as

$$1/(Q.Q_1) - 1/(Q.Q_3) = 1/(Q_1.Q_2) + 1/(Q.Q_2)$$

$$\text{Or, } (1/Q)(Q_3-Q_1)/(Q_1.Q_3) = (1/Q_2)(Q-Q_1)/(Q.Q_1) \tag{6}$$

Since Q, Q_1, Q_2, Q_3 are in Arithmetic Progression,

$$Q_1 - Q = Q_2 - Q_1 \quad \text{and} \quad Q_3 - Q_2 = Q_2 - Q_1 \tag{7}$$

Using the equalities of eq. (7), we can write

$$Q_3 - Q_1 = (2Q_2 - Q_1)-Q_1 = 2(Q_2 - Q_1) = 2(Q_1 - Q)$$

Substituting for $(Q_3 - Q_1)$ in eq. (6), we obtain

$$2(Q_1 - Q)/(Q.Q_1.Q_3) = (Q-Q_1)/(Q.Q_1.Q_2) \quad \text{Or,} \quad 2/Q_3 = -1/Q_2$$

$$\text{i.e., } Q_3(x) + 2Q_2(x) = 0,$$

which is exactly the sūtra formula $L+2P = 0$.

In addition to its mathematical aspects, Vedic Mathematics has a number of educational values. The first of these is : it gives insight into the process of mathematical computations, and makes one think and create. In this age of computers and automata, when the classroom use of calculators is becoming widespread, the need for thought, while working with numeral methods, is critically important for teachers and students alike. One must not forget, that while teaching, one teaches not only a specific numerical technique expressed in a specific programming language, but also a method of thinking about and formulating the solution of problems. Second, Vedic Mathematics gives a greater sense of mathematical involvement, and the students are better prepared intellectually, while at the same time providing an efficient and accurate method of solving mathematical problems. There is yet another way Vedic Mathematics can be made useful in the field of education. Let scholars follow the current methods of calculation; but while checking the results, they can follow the sūtra methods. This would provide an independent and faster method of verifying the results with a little or no change in the existing academic curriculae. It is my feeling that teaching of materials like these in a course on numeral computations will prove to be an extremely interesting and fruitful excercise. Besides, teaching of a course on Vedic Mathematics will not require prior experience in the subject.

Finally, success of Vedic Mathematics provides us with enough motivation to examine, in depth, the sūtra method as an alternative way (code language) of expressing mathematical terminology because it is simple, more compact than and as accurate as the symbolic method that is currently in use[34].

References

1. Bhāratī Kṛṣṇa Tīrthajī Mahārāja, Jagadguru Swāmī Śrī, Vedic Mathematics or Sixteen Simple Mathematical Formulae from the Vedas For One-line Answers to all Mathematical Problems, edited by Agrawala, Hindu Vishwavidyālaya Nepal Rājya Sanskrit Granthamālā Series, Volume 10, Banāras Hindu University, Vārāṇāsi, 1965.

2. Jain, N.K., History of Science and Scientific Method, Oxford & IBH Publishing Co., New Delhi, 1982, pp. 67-68.

3. Das, S., The Sūtra Method of Solving Algebraic Equations and Finding Sums of Fraction-Additions, Report BARC/1991/E/016, Bhabha Atomic Research Centre, Bombay, India, 1991, p. 1.

4. Gaṇitanand, In the name of Vedic Mathematics, Gaṇita-Bhāratī, 10, Nos. 1-4, 75-78, 1988.

5. Bhāratī Kṛṣṇa Tīrthajī Mahārāja, "Vedic Mathematics" p. x.

6. Radhakrishnan, S., Religion and Culture, Hind Pocket Books (P) Ltd., Delhi, 1968, pp. 22-50.

7. Datta, Bhagavada, Extraordinary Scientific Knowledge in Vedic Works, Arya Samaj, Diwan Hall, Delhi, 1963, p. 2.

8. Shastry, Vijayendra Ramakrishna, Bulletin of the National Institute of Sciences of India, 21, 94-104, 1983.

9. Nicholas, A.P., Review of the book "A Peep into Vedic Mathematics" by B.R. Baliga, Gaṇita-Bhāratī, 6, Nos. 1-4, 37-38, 1984.

10. Lal, B.B., Vedic Mathematics-mathematical calculations based on the Vedic sūtras and on the Līlavatī, Indian Journal of History of Science, 24(3), 161-162, 1989.

11. Das, S., Computational algorithms and Vedic Mathematics, Society and Science, 3, 21-37, 1980.

12. Das, S., Further Studies on Bhāratī Kṛṣṇa's Sūtra Method of Solving Algebraic Equations, Report BARC/1993/E/034, Bhabha Atomic Research Centre, Bombay, India, 1991, pp. 2-5.

13. Sastri, C. Sankara Rama (Ed.), Aṣṭādhyāyīsūtrapāṭha of Pāṇini, The Sri Balamanorama Press, Mylapore, Madras, 1937.

14. Jijñāsu, Brahmadatta (Ed.), Pāṇinīya Aṣṭādhyāyīsūtrapāṭhaḥ (in Hindi), Ramlal Kapur Trust, Amritsar, Punjab, 1964.

15. Mohan, Brij, Bulletin of the National Institute of Sciences of India, 21, 253-254, 1963.

16. Shukla, Kripa Shankar (Ed.), Āryabhaṭīya of Āryabhaṭa with the commentary of Bhāskara I and Someśvara, Indian National Science Academy, New Delhi, 1976, p. 109.

17. Hall, H.S. and Knight, S.R., Higher Algebra, Macmillan & Co. Ltd., London, 1957, p. 313.

18. Das, Varsha, Vivekananda Institute of Technology, Sindhi Society, Chembur, Bombay, India (Private Communication).

19. Stauffer, D., Hehl F.W., Winkelmann V. and Zabolitzky J.G., Computer Simulation and Computer Algebra, Springer-Verlag, Berlin and Heidelberg, 1988.

20. Haken, H., (Ed.), Neural and Synergetic Computers, Springer-Verlag, Berlin and Heidelberg, 1988.

21. Harper, David, Wooff Chris and Hodgkinson David, A guide to Computer Algebra Systems, John Wiley & Sons, Chichester and New York, 1991.

22. Schmid, Erich W., Spitz, Gerhard and Losch, Wolfgand, Theoretical Physics on the Personal Computers, Springer-Verlag, Berlin, 1990.

23. Taylor, Cyrus, Mathematics in the classroom, Computers in Physics Jan/Feb, 16-21, 1991.

24. Dill, Dan, Implementing matrick mechanics in Mathematica: determination of Clebsch-Gordan coefficients by matrix diagonalization, Computers in Physics, Nov/Dec, 616-624, 1991.

25. Baliga, B.R., A Peep Into Vedic Mathematics, Published privately by the Author, Trichur, India, 1979.

26. Mehta, P.N., A Study of Vedic Mathematical Principles and Their Validity in Relation to Modern Mathematics, M.Phil Dissertation, South Gujarat University, 1981.

27. Bhatnagar, P.L., Mathematics Teacher, Vol. X, 83-111, 1976.

28. Bhanu Murthy, T.S., A modern introduction to Ancient Indian Mathematics, Wiley Eastern Limited, New Delhi, 1992, pp. 59-101.

29. Bag, Amulya Kumar, Review of the book "Vedic Mathematics or 'SIXTEEN SIMPLE MATHEMATICAL FORMULAE FROM THE VEDAS' ", Indian Journal of History of Science, 3, No. 2, 59-60, 1968.

30. Gupta, R.C., Review of the book "Vedic Mathematics or Sixteen Simple Mathematical Formulae from the Vedas", Indian Journal of History of Science, 18, 222-225, 1983.

31. Kapur, J.N., The So-called Vedic Mathematics, Mathematical Education, 5, No. 4, 200-204, April-June 1989.

32. Sharma, Śri Ananta Tripāthī, Upaniṣat-Saṅgraha, Part 1 (in Oriya), Published by the author, Shiromaṇi Press, Berhampur, Orissa, 1960, pp. (1)-(2).

33. Ray, Saradaranjan, Kalidasa's Abhijñāna-Śakuntalam, Published by K. Ray, Vivekananda Road, Calcutta, 1959, pp. 1-4 of Appendix.

34. Ramanna, R., Sanskrit and Science, Bhavan's Book University, Bombay, 1984, p. 22.

Appendix 1

THE SŪTRAS (FORMULAE)

	Original in Devanāgarī	Roman Transliteration
1.	एकाधिकेन पूर्वेण	ekādhikena pūrveṇa
2.	निखिलं नवतश्चरमं दशतः	nikhilaṃ navataścaramaṃ daśataḥ
3.	ऊर्ध्वतिर्यग्भ्याम्	ūrdhva-tiryagbhyāṃ
4.	परावर्त्य योजयेत्	parāvartya yojayet
5.	शून्यं साम्यसमुच्चये	śūnyaṃ sāmya samuccaye

6.	(आनुरूप्ये) शून्यमन्यत्	(ānurūpye) śūnyam anyat
7.	संकलनव्यवकलनाभ्याम्	saṅkalana-vyavakalanābhyām (also a corollary)
8.	पूरणापूरणाभ्याम्	pūraṇāpūraṇābhyām
9.	चलनकलनाभ्याम्	calana-kalanābhyām
10.	यावदूनम्	yāvadūnam
11.	व्यष्टिसमष्टिः	vyaṣṭisamaṣṭiḥ
12.	शेषाण्यङ्केन चरमेण	śeṣāṇyaṅkena carameṇa
13.	सोपान्त्यद्वयमन्त्यम्	sopānty-advayamantyam
14.	एकन्यूनेन पूर्वेण	ekanyūnena pūrveṇa
15.	गुणितसमुच्चयः	guṇitasamuccayaḥ
16.	गुणकसमुच्चयः	guṇakasamuccayaḥ

Appendix 2

THE UPA(SUB)-SŪTRAS (COROLLARIES)

	Original in Devanāgarī	Roman Transliteration
1.	आनुरूप्येण	ānurūpyeṇa
2.	शिष्यते शेषसंज्ञः	śiṣyate śeṣasaṃjñaḥ
3.	आद्यमाद्येनान्त्यमन्त्येन	ādyam ādyenāntyam antyena
4.	केवलैः सप्तकं गुण्यात्	kevalaiḥ saptakaṃ guṇyāt
5.	वेष्टनम्	veṣṭanaṃ
6.	यावदूनं तावदूनम्	yāvadūnaṃ tāvad ūnaṃ
7.	यावदूनं तावदूनीकृत्य वर्गं च योजयेत्	yāvadūnaṃ tāvad ūnīkṛtya vargañca yojayet
8.	अन्त्ययोर्दशकेऽपि	antyayor daśake'pi
9.	अन्त्ययोरेव	antyayor eva
10.	समुच्चयगुणितः	samuccaya-guṇitaḥ
11.	लोपनस्थापनाभ्याम्	lopana-sthāpanābhyām
12.	विलोकनम्	vilokanam
13.	गुणितसमुच्चयः समुच्चयगुणितः	guṇita-samuccayaḥ samuccaya-guṇitaḥ

Appendix 3

Solutions of Sample Problems using the Current Methods

Problem 4

The equation is (x-coefficient rational number) :

$$2/[(13x/3)-1] + 1/[(13x/3)+3] = 0$$

Or, $\{2[(13x/3)+3] + 1[(13x/3)-1]\}/[(13x/3)-1][(13x/3)+3] = 0$

Or, $2(13x+9) + (13x-3)/3 = 0$

Or, $26x+18+13x-3 = 0$

Or, $39x + 15 = 0$ Or, $13x + 5 = 0 \Rightarrow x = -5/13$

Problem 6

The equation is (x-coefficient irrational) :

$$2/(\sqrt{3}x-1) + 5/(\sqrt{2}x+7) = (2\sqrt{2}+5\sqrt{3})/(\sqrt{6}x+9)$$

Or, $[2(\sqrt{2}x+7) + 5(\sqrt{3}x-1)]/(\sqrt{3}x-1)(\sqrt{2}x+7) = (2\sqrt{2}+5\sqrt{3})/(\sqrt{6}x+9)$

Simplifying and cross-multiplying

$$[x(2\sqrt{2}+5\sqrt{3}) + 9]\,[\sqrt{6}x+9] = (2\sqrt{2}+5\sqrt{3})\,(\sqrt{6}x^2+7\sqrt{3}x-\sqrt{2}x-7)$$

Or, $x^2\,(2\sqrt{12}+5\sqrt{18}) + 9\sqrt{6}x + 9x(2\sqrt{2}+5\sqrt{3}) + 81$

$$= (2\sqrt{12}+5\sqrt{18})x^2 + (14\sqrt{6}+105)x - (4+5\sqrt{6})x - 14\sqrt{2} - 35\sqrt{3}$$

Or, $x(9\sqrt{6}+18\sqrt{2}+45\sqrt{3}) + 81 = x(9\sqrt{6}+101) - 14\sqrt{2} - 35\sqrt{3}$

Cancelling out the common term and rearranging,

$$x(18\sqrt{2}+45\sqrt{3}-101) = -(14\sqrt{2} + 35\sqrt{3} + 81)$$

Therefore, $x = (14\sqrt{2}+35\sqrt{3}+81)/(101-18\sqrt{2}-45\sqrt{3})$

Problem 9

The simultaneous equations are (the x and y-coefficients are either integers or rational numbers or irrational numbers) :

$$\sqrt{19}x + (5/2)y = 7 \} \quad (1)$$

$$3x + \sqrt{13}y = 4/9\} \quad (2)$$

Multiply eq. (1) by 3 and eq. (2) by $\sqrt{19}$, and subtract. We obtain,

$$(5/2)3y - \sqrt{13}\,\sqrt{19}y = 21-(4/9)\sqrt{19}$$

Or, $y\,[(15 - 2\sqrt{247})/2] = (189-4\sqrt{19})/9$

Therefore, $y = (2/9)(189-4\sqrt{19})/(15-2\sqrt{247})$

Substituting this value of y in eq. (2), we get

$$3x + \sqrt{13}(2/9)(189-4\sqrt{19})/(15-2\sqrt{247}) = 4/9$$

Or, $3x = (4/9) - (2/9)\,(189\sqrt{13}-4\sqrt{247})/(15-2\sqrt{247})$

$$= [4(15-2\sqrt{247}) - 2(189\sqrt{13}-4\sqrt{247})]/9(15-2\sqrt{247})$$

$$= (60-378\sqrt{13})/9(15-2\sqrt{247}) = 6(10-63\sqrt{13})/(15-2\sqrt{247})9$$

Therefore, $x = (2/9)\,(10-63\sqrt{13})/(15-2\sqrt{247})$

Problem 13

The simultaneous equations are (x-coefficients & constants integers)

$$499x + 172y = 212 \quad (1)$$

$$9799x + 387y = 477 \quad (2)$$

Multiply eq. (1) by 387 and eq. (2) by 172, and subtract.

499	387
x387	x212
3493	774
3992	387
1497	774
193113	82044

$$\begin{array}{r} 9799 \\ \times 172 \\ \hline 19598 \\ 68593 \\ 9799 \\ \hline 1685428 \end{array} \qquad \begin{array}{r} 477 \\ \times 172 \\ \hline 954 \\ 3339 \\ 477 \\ \hline 82044 \end{array}$$

Therefore, we get

$$\begin{array}{rl} 193113x & = 82044 \\ 1685428x & = 82044 \\ - & - \\ \hline -\ 1492315x = 0 & \Rightarrow \quad x = 0 \end{array} \qquad \begin{array}{r} 1685428 \\ 193113 \\ \hline 1492315 \end{array}$$

Substituting this value of x in eq. (1), we get

$$499 \times 0 + 172y = 212 \quad \text{Or,} \quad 172y = 212 \quad \Rightarrow \quad y = 212/172 = 53/43$$

Therefore, the solution is (0, 53/43).

Problem 14

The three equations in three variables are :

$$97x + ay + 43z = am \qquad (1)$$

$$49979x + by + (p+q)z = bm \qquad (2)$$

$$49(a-d)^3x + cy + (m-n)^3z = cm \qquad (3)$$

Set $P = p+q$, $Q = (a-d)^3$, $R = (m-n)^3$. Multiply eq. (1) by 49979 and eq. (2) by 97 and subtract eq. (2) from eq. (1).

We get $(49979a-97b)y + (43x49979-97P)z = (49979a-97b)m$ (4)

Again, multiply eq. (2) by 49Q and eq. (3) by 49979 and subtract eq. (3) from eq. (2). We get

$$(49bQ-49979c)y + (49PQ-49979R)z = (49bQ-49979c)m \qquad (5)$$

Now multiply eq. (4) by (49bQ–49979c) and eq. (5) by (49979a–97b) and subtract. We obtain

$$[(49bQ-49979c)(43\times49979-97P) - (49979a-97b)(49PQ-49979R)]z$$

$$= [(49bQ-49979c)(49979a-97b) - (49979a-97b)(49bQ-49979c)]m = 0$$

Therefore, $z = 0$

Substituting for z in eq. (4), we get

$$(49979a-97b)y = (49979a-97b)m \quad \Rightarrow \quad y = m$$

Substituting for y and z in eq. (1), we obtain

$$97x + am = am \quad \Rightarrow \quad x = 0$$

Therefore, the solution is (0, m, 0).

Problem 15

The simultaneous equations are (rational or irrational coefficients) :

$$(23/2)x - \sqrt{41}y = 105\} \qquad (1)$$

$$\sqrt{41}x - (23/2)y = 99\} \qquad (2)$$

Multiply eq. (1) by $\sqrt{41}$ and eq. (2) by 23/2 and subtract. We get,

$$y[(23/2)^2-41] = 105\sqrt{41}-(2277/2)$$

Or, $[(529-164)/4]y = (210\sqrt{41}-2277)/2$

$$\Rightarrow \qquad y = (420\sqrt{41}-4554)/365$$

Substituting for y in eq. (1), we obtain

$$(23/2)x - \sqrt{41}(420\sqrt{41}-4554)/365 = 105$$

Or, $(23/2)x = 105+\sqrt{41}(420\sqrt{41}-4554)/365$

$$= (105\times365+420\times41-4554\sqrt{41})/365$$

Therefore, $x = (38325+17220-4554\sqrt{41})2/(23\times365)$

$$= 2(55545-4554\sqrt{41})/(23x365) = 2(2415-198\sqrt{41})/365$$

$$= (4830-396\sqrt{41})/365$$

Problem 16

The equation is (integral co-efficients) :

$$1/(3x^2+5x+2) + 1/(5x^2+8x+3) = 1/(7x^2+11x+4) + 1/(15x^2+19x+6)$$

Or, $1/(3x^2+3x+2x+2) + 1/(5x^2+5x+3x+3)$

$$= 1/(7x^2+7x+4x+14) + 1/(15x^2+10x+9x+6)$$

Or, $1/[3x(x+1)+2(x+1)] + 1/[5x(x+1)+3(x+1)]$

$$= 1/[7x(x+1)+4(x+1)] + 1/[5x(3x+2)+3(3x+2)]$$

Or, $1/(x+1)(3x+2) + 1/(5x+3)(x+1) = 1/(7x+4)(x+1) + 1/(5x+3)(3x+2)$

Or, $(5x+3+3x+2)/(x+1)(3x+2)(5x+3)$

$$= (15x^2+10x+9x+6+7x^2+7x+4x+4)/(7x+4)(x+1)(5x+3)(3x+2)$$

Or, $(8x+5)(7x+4) = 22x^2+30x+10$ Or, $56x^2+32x+35x+20-22x^2-30x-10 = 0$

Or, $34x^2 + 37x + 10 = 0$. This is a quadratic equation in x.

Therefore, $x = [-37 \pm \sqrt{(37^2-4x34x10)}]/(2x34)$

$$= [-37 \pm \sqrt{(1369-1360)}]/68 = (-37\pm3)/68=-40/68 \text{ or } -34/68$$

Therefore, $x = -10/17$ or $-1/2$

Problem 17

The equation is (x-coefficients integral or rational) :

$$[(x/2)+\sqrt{5}]^2/[(x/3)+\sqrt{7}]^2 = (9x+36\sqrt{5})/(4x+24\sqrt{7})$$

Or, $[(x+2\sqrt{5})^2/4][9/(x+3\sqrt{7})^2] = (9x+36\sqrt{5})/(4x+24\sqrt{7})$

Cross-multiplication gives

$$9(x^2+4\sqrt{5}x+20)(4x+24\sqrt{7}) = 4(x^2+6\sqrt{7}+63)(9x+36\sqrt{5})$$

Or, $9(4x^3+24\sqrt{7}x^2+16\sqrt{5}x^2+96\sqrt{35}x+80x+480\sqrt{7})$

$= 4(9x^3+36\sqrt{5}x^2+54\sqrt{7}x^2+216\sqrt{35}x+567x+2268\sqrt{5})$

Or, $216\sqrt{7}x^2+144\sqrt{5}x^2+864\sqrt{35}x+720x+4320\sqrt{7}$

$= 144\sqrt{5}x^2+216\sqrt{7}x^2+864\sqrt{35}x+2268x+9072\sqrt{5}$

Or, $1548x = (4320\sqrt{7}-9072\sqrt{5})$. Simplifying, we get

$x = (120\sqrt{7} - 252\sqrt{5})/63$

Problem 19

Sum to n terms the series

$1/(1+x)(1+2x) + 1/(1+2x)(1+3x) + 1/(1+3x)(1+4x) + \dots\dots\dots$

If we denote the series by

$u_1+u_2+u_3+\dots\dots+u_n$, we have

$u_1 = [1/(1+x) - 1/(1+2x)]/x$

$u_2 = [1/(1+2x) - 1/(1+3x)]/x$

$u_3 = [1/(1+3x) - 1/(1+4x)]/x$

....

$u_n = [1/(1+nx) - 1/\{1+(n+1)x\}]/x$

Therefore, by addition, we get the sum

$S_n = [1/(1+x) - 1/\{1+(n+1)x\}]/x$

$= (1+nx+x-1-x)/x[1+x][1+x][1+(n+1)x]$

$= n/[1+x][1+(n+1)x]$

Problem 20

Find the sum of

$[1/(1+x)(1+ax)]+[a/(1+ax)(1+a^2x)]+[a^2/(1+a^2x)(1+a^3x)]+\dots\dots$ to n terms.

The n^{th} term (u_n) is $= [a^{n-1}/(1+a^{n-1}x)(1+a^n x)]$

$$= (A/(1+a^{n-1}x) + B/(1+a^n x) \text{ suppose;}$$

Therefore, $\quad a^{n-1} = A(1+a^n x) + B(1+a^{n-1}x)$

By putting $1+a^{n-1}x = 0,$ (1)

We obtain, $A = a^{n-1}/(1+a^n x)$ (2)

From eq. (1), $x = -1/a^{n-1}$. Substituting for x in eq. (2),

$$A = a^{n-1}/[1-(a^n/a^{n-1})] = a^{n-1}/(1-a)$$

Similarly, by putting $1+a^n x$ equal to zero, we get

$$B = a^{n-1}/(1+a^{n-1}x) = a^{n-1}/[1+(a^{n-1})(-1/a^n)]$$

$$= a^{n-1}/(1-a^{-1}) = -a^n/(1-a)$$

Therefore,

$$u_n = [a^{n-1}/(1-a)(1+a^{n-1}x)] - [a^n/(1-a)(1+a^n x)] \quad \text{and}$$

$$u_1 = [1/(1+x) - a/(1+ax)]/(1-a)$$

$$u_2 = [a/(1+ax) - a^2/(1+a^2x)]/(1-a)$$

....

$$u_n = [a^{n-1}/(1+a^{n-1}x)] - [a^n/(1+a^n x)]/(1-a)$$

Adding all the terms, we get

$$S_n = [1/(1+x) - a^n/(1+a^n x)]/(1-a) = (1-a^n)/(1-a)(1+x)(1+a^n x)$$

Problem 21

Sum of the first four terms (proper fractions) is

$$S_4 = 3/70 + 9/190 + 27/874 + 99/6670$$

L.C.M. of the denominators

2	70,	190,	874,	6670
5	35,	95,	437,	3335
19	7,	19,	23,	667
23	7,	1,	23,	29
	7,	1,	1,	29

Therefore, S_4 = (3x19x23x29 + 9x23x7x29 + 27x5x7x29 + 99x19x7)/(2x5x19x23x7x29)

= (57x23x29 + 63x23x29 + 27x35x29 + 99x133)/(10x19x23x7x29)

57	63	35	133
x23	x23	x27	x99
171	189	245	1197
114	126	70	1197
1311	1449	945	13167

1311	1449	945
x29	x29	x29
11799	13041	8505
2622	2898	1890
38019	42021	27405

S_4 = (38019+42021+27405+13167)/(10x19x23x7x29)

= 120612/(10x19x23x7x29) = 60306/(5x19x23x7x29)

19) 60306 (3174	23) 3174 (138	38019
57	23	42021
33	87	27405
19	69	13167
		120612

140	184
133	184
——	——
76	000
76	
——	
00	

Hence, $S_4 = 138/(5x7x29) = 138/(35x29) = 138/3015$