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USE OF PERMUTATIONS AND COMBINATIONS IN INDIA

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Interest of the Hindus in the subject of permutations and combinations originated in connection with the variation of the Vedic metres in a very early age. There are specific rules for the calculation of the variation of metres in the *Chandaḥ-sûtra* of Piṅgala (before 200 BC). Permutations and combinations seem to have been subjects of such a fascinating study for the Hindus that they applied the ideas about them in various other spheres of life, e.g. architecture, music, medicine and astrology. Application of the principles of permutations and combinations is also found in the canonical literature of the Jainas in the study of philosophical categories. The present article aims at giving an account of the various uses of permutations and combinations in Indian literature.

EARLY INTEREST IN THE SUBJECT

The Hindu interest in the subject of permutations and combinations began in a very early age, first probably in connection with the variation of the Vedic metres and philosophical categories[1]. In the *Chandaḥ-sûtra* ("Rules of the Metre") of Piṅgala, a work on Vedic metres, written before 200 BC, we find specific rules for computation of the possible number of variations of even, semi-even, and uneven metres in a group with a specified number of long and short syllables in a quarter of a verse. In the *Nāṭya-śāstra* of Bharata Muni[2] are stated the number of variations of even metres having six to 26 syllables in a quarter of a verse. In the classical treatise on Hindu medicine by Suśruta, called *Suśruta-saṁhitā*[3], written about 600 BC, the total number of combinations that can be made out of six savours taking one, two, three,..., five and all at a time is found to have been correctly stated as 63. The early canonical literature of the Jainas (500-300 BC)[4] abounds in instances of speculation about the different sub-categories that can arise out of a fixed number of fundamental philosophical categories by the combinations of one, two, or more of them at a time. There are also similar calculations of the groups that can be formed out of the different instruments of senses, of the selections that can be made out of a number of males, females, and eunuchs or of permutations and combinations of various other things. The principles of the subject seem to have appealed to the Hindu mind and are found to have been applied in various spheres, such as astrology, perfumery, architecture, and music, besides those mentioned above. Thus, Bhāskara II (1150) observes: "It serves in prosody, for those versed therein, to find the variations of metres; in architecture to compute the changes in apertures, etc. (of a building); (in music), the scheme of musical permutations; and in medicine, the combination of different savours"[5].

TERMINOLOGY

The oldest Hindu names for the subject of permutations and combinations are *vikalpa* (lit. "alternatives", "variations") and *bhaṅga* (lit. "poses"). Both these terms occur in the early canonical works of the Jainas (500-300 BC). The term *vikalpa* can be traced still earlier in the *Suśruta-saṁhitā* (*c.* 600 BC). Brahmagupta (628) calls it *Chandaściti* ("piling of metres"[6]), obviously because it originated, as has been stated above, in connection with the variation of Vedic metres. This name appears in later works also. Mahāvīra (850) calls combinations by the term *yutibheda* ("variations of combinations"[7]) and Śrīdhara (*c.* 750) and Bhaskara II (1150) by the term *bheda* or *vibheda* ("variation") only[8]. Bhāskara II introduces the names *aṅkapāśa* ("concatenation of numbers") and *gaṇita-pāśa*[9] for permutations. Nārāyaṇa (1356) has used the term *aṅka-pāśa*[10] to denote the whole subject of permutations and combinations. The Hindu expressions corresponding to the modern "taken one at a time", "taken two at a time", etc. are *ekaka-saṁyoga* (lit. "one-combination"), *dvika-saṁyoga* ("two-combination"), etc. These terms occur from the *Suśruta-saṁhitā* onwards. Other terms used in that sense are *eka-vikalpa* ("one variation"), *dvi-vikalpa* ("two variation"), etc.

SUŚRUTA'S RULES FOR COMBINATIONS

Suśruta (*c.* 600 BC)[11] states that the number of combinations of six savours — sweet, acid, saline, pungent, bitter and astringent — taken two at a time is 15. He seems to have arrived at it by writing down all the combinations exhaustively. For he observes: "On making two combinations in successive way, those beginning with sweet are found to be 5 in number; those beginning with acid are 4; those with saline 3; those with pungent 2; bitter and astringent make 1 combination". He then presents the actual combinations thus: sweet-acid, sweet-saline, sweet-pungent, sweet-bitter, sweet-astringent; acid-saline; acid-pungent; acid-bitter; acid-astringent; saline-pungent, saline-bitter, saline-astringent; pungent-bitter, pungent-astringent; and bitter-astringent. In the same way, Suśruta finds the number of 3-combinations to be 20; 4-combinations 15; 5-combinations 6; and 6-combinations 1. Thus, there are $6+15+20+15+6+1=63$ different combinations in all.

JAINA CANONICAL WORKS

In the early canonical works of the Jainas (500-300 BC), we find the results which correspond to

$$^{n}C_1 = n,\ ^{n}C_2 = \frac{n(n-1)}{1.2},\quad ^{n}C_3 = \frac{n(n-1)(n-2)}{1.2.3},\ \ldots$$

After stating the results in case of $n=1, 2, 3, 4$, the *Bhagavatī-sūtra* observes: "And in this way 5, 6, 7, ..., 10, etc. numerable, innumerable, or infinite number of things may be mentioned. Forming one-combinations, two-combinations, three-combinations, and so on, ten-combinations, eleven-combinations, twelve-combinations, etc., as the successive combinations are formed, all of them should be considered[12]."

VARĀHAMIHIRA'S RULE

To find the number of combinations of n unlike things taken 1, 2, 3, ... at a time successively, Varāhamihira (*d.* 587) gives the following rule:

"They say that the number (of combinations) is obtained by (writing down the natural numbers 1, 2, 3, etc. up to the total number of things, one above the other, and) adding the preceding number to the succeeding one (in succession) and rejecting the last number [13]". The commentator Bhaṭṭotpala (966) has explained the process clearly by taking 16 different things. We reproduce from him the following scheme for it:

16			
15	120		
14	105	560	
13	91	455	1820
12	78	364	1365
11	66	286	1001
10	55	220	715
9	45	165	495
8	36	120	330
7	28	84	210
6	21	56	126
5	15	35	70
4	10	20	35
3	6	10	15
2	3	4	5
1	1	1	1

(The topmost number in the first column gives ${}^{16}C_1$, that in the second column gives ${}^{16}C_2$, that in the third column ${}^{16}C_3$, and that in the fourth column ${}^{16}C_4$. To get ${}^{16}C_5$ and others, the process of forming the successive columns should be continued further on.)

While dealing with the manufacture of perfumes in his *Bṛhat-saṁhitā*, Varāhamihira says:

"An immense number of perfumes can be made out of 16 ingredients, if every 4 of them are combined at will in one, two, three, and four proportions..... The total number of these perfumes will be 174720. Each substance (of a group of four) taken in one proportion being combined with the other three, taken in two, three, and four proportions, gives rise to 6 perfumes; and so it does, when taken in two, three or four proportions. One substance associated with a group of four substances (thus) gives rise to 24 perfumes; and in the same way the remaining three substances (of that group) (also give rise to 24 perfumes). The total of all these is 96. Now when 16 substances are divided into separate groups of 4 each, there arise 1820 such groups. Since each group of four gives rise to 96 varieties (of perfumes), therefore, that number (i.e., 1820) should be multiplied by 96. The number (resulting from this product) is the (total) number of perfumes[14]".

In another place, Varāhamihira states, "There are 31 varieties of *Anaphā-yoga* and *Sunaphā-yoga* each, and 180 of *Durudharā-yoga*[15]." Now it has been defined that an *Anaphā-yoga* occurs when one or more of the five planets, Mars, Mercury, Jupiter, Venus and Saturn, occupy the twelfth house from the Moon; in *Sunaphā-yoga*, a similar occurrence takes place in the second house from the Moon; and in the *Durudharā-yoga*, the planets occupy both these houses. Hence, we get

$$^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 5 + 10 + 10 + 5 + 1 = 31$$

$$\begin{aligned} & {}^5C_1({}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) + {}^5C_2({}^3C_1 + {}^3C_2 + {}^3C_3) + {}^5C_3({}^2C_1 + {}^2C_2) + {}^5C_4({}^1C_1) \\ &= 75 + 70 + 30 + 5 \\ &= 180. \end{aligned}$$

Brahmagupta (628) has devoted one full chapter (20th) of his treatise on astronomy, the *Brāhma-sphuṭa-siddhānta*, to the treatment of variation of metres. But on account of faulty readings, it has not been possible to make proper sense out of it.

Śrīdhara's Rule

To find the number of combinations of the six savours, taken one, two, three, ..., five, and all at a time, Śrīdhara gives the following rule:

"Writing down the numbers beginning with one and increasing by one up to the (given) numbers of savours, in the inverse order, divide them by the numbers beginning with one and increasing by one in the regular order, and then multiply successively by the preceding (quotient) the succeeding one[16]".

Thus, writing the numbers of the savours 1, 2, 3, 4, 5, 6 in the inverse order and dividing them by the same numbers in the regular order, we get

$$\frac{6}{1}, \quad \frac{5}{2}, \quad \frac{4}{3}, \quad \frac{3}{4}, \quad \frac{2}{5}, \quad \frac{1}{6}.$$

Performing the successive multiplication by the preceding quotient of the succeeding one, we get

$$\frac{6}{1}, \ \frac{6}{1}\times\frac{5}{2}, \ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}, \ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}, \ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}\times\frac{2}{5}, \ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}\times\frac{2}{5}\times\frac{1}{6}.$$

These are the values of $^6C_1$, $^6C_2$, $^6C_3$, ..., $^6C_6$ respectively.

Mahāvīra's Rule

To find the number of combinations of unlike things, Mahāvīra gives the following general rule:

"Set down the numbers beginning with unity and increasing by one, up to the (given) number (of things) in the regular and inverse orders in upper and lower rows respectively. The product of the numbers (in the upper row) taken right-to-left-wise being divided by the product of the (corresponding) numbers (in the lower row) taken in the same way, the quotient gives the result[17]".

That is to say, if there be n things, we shall have the arrangement

$1, 2, 3, \ldots\ldots, n-r, n-r+1, \ldots\ldots, n-2, n-1, n$

$n, n-1, n-2, \ldots\ldots, r+1, r, \ldots\ldots, 3, 2, 1.$

Then says Mahāvīra

$$^{n}C_{r} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3.\ldots\ldots r}$$

It is perhaps noteworthy that one of the illustrative examples given by both Śrīdhara and Mahāvīra is the same as that given by Suśrutā[18]. It appears also in Bhāskara II's *Līlāvatī*[19], and Nārāyaṇa's *Gaṇita-kaumudī*[20].

Śrīśaṅkara's Rule

Bhaṭṭotpala (966) has quoted the following rule from another writer, probably Bhaṭṭa Śrīśaṅkara, of whom we know very little now:

"Write down (the natural numbers) in the reverse way and below them in the regular way. Multiply the numbers (in the two rows) taken in the regular way and divide the product from the upper row by that from the lower[21]".

So, the scheme in this case is

$n, n-1, n-2, \ldots\ldots, n-r+1, n-r, \ldots\ldots, 3, 2, 1$

$1, 2, 3, \ldots\ldots, r, r+1, \ldots\ldots, n-2, n-1, n.$

Bhāskara II's Rule

Bhāskara II (1150) says:

"Divide the numbers from one upwards, increasing by unity, set down in the inverse order, by the same (arithmeticals) written in the regular order. The first quotient, the second multiplied by the first, the next multiplied by that, and so on, give the combinations by one, two, three, etc. This is the general rule[22]".

An example from Bhāskara II:

"A pleasant, spacious and elegant palace, constructed by a skilful architect for the landlord, has eight apertures in it. Tell me the number of combinations of them formed by taking one, two, three, etc. (at a time)."

The total number of combinations

$$= {}^8C_1 + {}^8C_2 + {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$$

$$= 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1$$

$$= 255.$$

EARLY RULE FOR PERMUTATIONS

In the early canonical works of the Jainas, we find copious instances of calculation of permutations yielding results corresponding to the modern formulae.

$${}^nP_1 = n,\ {}^nP_2 = n(n-1),\ {}^nP_3 = n(n-1)(n-2),\ \text{etc.}$$

But the earliest mention of a rule for finding the number of permutations of n things taken all at a time is found in the *Anuyogadvāra-sūtra*, a canonical work written before the beginning of the Christian era. It says:

"What is the direct arrangement? *Dharmāstikāya, Adharmāstikāya, Ākāśāstikāya, Jīvāstikāya, Pudgalāstikāya* and *Addhāsamaya* — this is the direct arrangement. What is the reverse arrangement? *Addhāsamaya, Pudgalāstikāya, Jīvāstikāya, Ākāśastikāya, Adharmāstikāya* and *Dharmāstikāya* — this is the reverse arrangement. What are the mixed arrangements? Form the series of numbers beginning with one and increasing by one up to six terms. The mutual products of these minus 2 will give the number of mixed arrangements[24]".

We have similar rules for 7, 10, 16, 24 or any variable number (*asaṁkhyeya*) of unlike things[25]. Thus, it was known that the number of permutations of n unlike things taken all at a time is

$$1.2.3\ldots(n-2)(n-1)n.$$

JINABHADRA GAṆI'S RULE

Jinabhadra Gaṇi (529-589) says:

"Multiply mutually the numbers beginning with one and increasing by one up to the numbers of terms (i.e., unlike things); then the product (will give the number of permutations)[26]".

A similar rule has been given by the commentator Śīlāṅka (862) from an unknown writer:

"Beginning with unity up to the number of terms, multiply continuously the (natural) numbers. The product should be known as the result (i.e. the total number) in the calculation of permutations (*vikalpagaṇita*)[27]".

Bhāskara II's Rules

To find the number of permutations of n unlike things taken all at a time, Bhāskara II (1150) gives a rule similar to those stated above:

"The product of the numbers beginning with and increasing by unity and continued up to the number of places will be the number of different permutations with all of the specified things[28]".

He then gives a rule for finding the permutations of n unlike things taking any variable number of them at a time.

"The product of the numbers from the total number of places and decreasing by unity, continued up to the last of the (variable) places gives the number of permutations of unlike things[29]".

That is to say, the number of r permutations of n dissimilar things will be

$$n(n-1)(n-2).....\text{up to r factors.}$$

Similar rules are given by Nārāyaṇa[30].

Permutations of Things Not All Different

To find the number of ways in which n things may be arranged amongst themselves, taking all at a time, when some of the things are alike, Bhāskara II gives the following rule:

"Find separately the number of permutations for as many places as are occupied by like digits; then divide by that the number of permutations calculated before (on the supposition that all the digits are unlike): the quotient will be the (required) number of permutations[31]".

A similar rule is given by Nārāyaṇa[32].

"That is to say, if p of the digits are alike of one kind, q of them are alike of a second kind, r of them are alike of a third kind, and the rest all different, then the number of permutations will be

$$\frac{n!}{p!\, q!\, r!},$$

n being the total number of places occupied by the digits (like and unlike).

Examples from Bhāskara II[33]

The different numbers that can be formed out of the digits 2, 2, 1, 1 are in all

$$\frac{4!}{2!\, 2!} = 6.$$

The various numbers that can be formed out of the digits 4, 8, 5, 5, 5 are altogether

$$\frac{5!}{3!} = 20.$$

Nārāyaṇa[34] states that when each of the n things is repeated, the number of r-permutations is n^r. As examples, he finds that with the digits 1 and 2 there can be formed as many as 2^6 or 64 numbers of six notational places each, and with the digits 1, 2 and 3 will be obtained 3^3 or 27 numbers of three notational places each[35].

Sum of Permutations

To find the sum of the numbers that can be formed by the permutations of some given digits, taken all at a time, Bhāskara II gives the following rule:

"That (the number of permutations) is divided by the number of digits and multiplied by their sum; the result being repeated according to the notational places (as many times as the number of digits) and added together will give the sum of the permuted numbers[36]".

This rule is equally applicable to both the cases when all the digits are unlike and when some of them are alike[37].

Illustrative examples from Bhāskara II[38]

(1) The numbers that can be formed by permutation of the eight digits 2, 3, 4, 5, 6, 7, 8, 9 are altogether

$$= 1.2.3.4.5.6.7.8.$$

$$= 40320$$

Now we have

$$\frac{40320}{8}(2+3+4+5+6+7+8+9)=221760;$$

also setting down 221760 eight times advanced forward one place each time and then adding together, we get

```
221760
 221760
  221760
   221760
    221760
     221760
      221760
       221760
-------------
2463999975360
```

Hence, the sum of the numbers obtained by permutation is 2463999975360.

(2) The number that can be formed by the digits 2,2,1,1 has been found to be equal to 6 altogether. Now, we get

$$\frac{6}{4}(2+2+1+1)=9;$$

and also

```
9
 9
  9
   9
----
9999
```

Hence, the required sum in 9999.

The *rationale* of the rule is as follows[39].

Case 1. Suppose there are n digits and all of them are *unlike*.

The number of permutations that can be formed with these digits is n!. Now consider any of the digits, say a. In (n − 1)! of the numbers a will be in the units' place; in as many cases it will be in the tens' place; and so on. The sum arising from a alone, since there are n digits in all,

$$= (n-1)!(10^{n-1}a + 10^{n-2}a + \ldots\ldots + 10a + a)$$

$$= (n!/n)\ (10^{n-1} + 10^{n-2} + \ldots\ldots + 10 + 1)a$$

Proceeding in the same way with the other digits and adding up the partial sums, we get the sum of all the numbers resulting from permutations of the digits

$$= (n!/n)\ (10^{n-1} + 10^{n-2} + \ldots\ldots + 10 + 1)\ (\text{sum of the digits})$$

Case 2. Suppose p of the digits to be alike and equal to k_1, q of them equal to k_2, r of them equal to k_3 and the rest unlike.

The number of permutations that can be made with these digits is

$$\frac{n!}{p!q!r!}$$

The number of cases in which k_1 is in the units' place is

$$\frac{(n-1)!}{(p-1)!q!r!}$$

In as many cases it is in the tens' place; and so on. Hence, the partial sum arising out of k_1 is

$$\frac{(n-1)!}{(p-1)!q!r!}\ (10^{n-1} + 10^{n-2} + \ldots + 10 + 1)k_1$$

In the same way, the partial sums arising from k_2 and k_3 are respectively

$$\frac{(n-1)!}{(q-1)!p!r!}\ (10^{n-1} + 10^{n-2} + \ldots\ldots + 10 + 1)k_2$$

$$\frac{(n-1)!}{(r-1)!p!q!}\ (10^{n-1} + 10^{n-2} + \ldots\ldots + 10 + 1)k_3$$

and the partial sum due to the unlike digits k_4, k_5,.....is, by Case 1

$$\frac{(n-1)!}{p!q!r!}\ (10^{n-1} + 10^{n-2} + \ldots\ldots + 10 + 1)\ (k_4 + k_5 + \ldots.)$$

Hence, the required sum of all the numbers is

$$\frac{n!}{n\ p!q!r!}\ (10^{n-1} + 10^{n-2} + \ldots\ldots + 10 + 1)\ (pk_1 + qk_2 + rk_3 + k_4 + k_5 + \ldots.)$$

$$= \frac{n!}{n\,p!q!r!}(10^{n-1}+10^{n-2}+\ldots\ldots+10+1)\text{ (sum of all the digits).}$$

BHĀSKARA II'S PROBLEM

Bhāskara II proposed an interesting problem: To find how many different numbers occupying a specified number of notational places can be formed out of digits having a definite sum. His solution is as follows:

"When the sum of the digits is fixed, divide the successive numbers beginning with that sum minus one, and decreasing by one, continued up to one less than the number of places, by one, two, etc. respectively. The variations of numbers will be equal to the product of those quotients. This rule is valid, it must be known, only when the sum of the digits is less than the specified number of notational places plus nine".[40]

Illustrative example from Bhāskara II[41]

The different numbers of 5 digits of sum 13 will be altogether

$$\frac{12}{1}\cdot\frac{11}{2}\cdot\frac{10}{3}\cdot\frac{9}{4} = 495.$$

REPRESENTATION

It has been noted before that the interest of the ancient Hindus in the subject of permutations and combinations was not of theoretical origin, but grew out of a concrete purpose. For that it was essential not only to know the number of possible variations but also, and in a greater degree, to have the actual variations. So, we find that as early as the time of the Jaina canonical works, distinct consideration was being made between *bhaṅga-samutkīrṇatā* ("Telling permutations or combinations", that is, "Enumeration of possible variations") and *bhaṅga-pradarśanatā* ("Representation of permutations and combinations"). In the early state of the subject even the number of variations in any given case very probably used to be determined by writing them all down exhaustively. But the latter was obviously a laborious task and was often liable to be in error if all the operations be not carried out in a systematic way. Such a systematic scheme of operations is technically called the *Loṣṭa-prastāra* ("Spreading out of marked objects"), apparently because in the beginning the permutations or combinations used to be formed out of any given number of things by laying out objects, probably clay pieces, marked with the tachygraphic abbreviations of the names of the various things.

REPRESENTATION OF COMBINATIONS

A scheme of writing down all the possible combinations formed out of a given number of unlike things is sufficiently clear from the descriptions of Suśruta. The same appears in the Jaina canonical works[42]. Varāhamihira's rule for that is as follows:

"Any one of the things taken optionally should be successively operated upon (by the rest); when that process is exhausted, the next (should be begun)"[43].

The operations implied have been explained at length by Bhaṭṭotpala with the help of specific instances. In this connection he has quoted a rule from Bhaṭṭa Śrīśaṅkara[44]. Jinabhadra Gaṇi (*c.* 550) also has a rule for the same[45].

Representation of Permutations

Śīlāṅka (862)[46] has quoted a rule from an ancient writer who is not known now, describing a systematic scheme of forming all the possible permutations out of a given number of unlike things:

"The total number of permutations should be divided by the last term, then the quotient by the rest. They should be placed successively by the side of the initial term in the calculation of permutations."

The rule appears to be cryptic, but Śīlāṅka has explained it clearly with the help of an illustrative example: To find the numbers that can be formed by using the digits 1, 2, 3, 4, 5, 6. It is as follows:

Let there be n number of things a_1, a_2, ..., a_n. Then the total number of permutations that can be formed out of them will be n!. The number of permutations which can have any particular thing, say a_1, for its initial digit (*ādi*) will be n!/n, that is, (n − 1)!. So, put a_1 in the beginning of (n − 1)! grooves and so on. Again amongst the first series of grooves, the number of sub-grooves that can have a_2 after a_1 will be (n − 1)!/(n − 1) or (n − 2)!. Place a_2 after a_1 in those sub-grooves. The number of sub-grooves that can have a_3 after a_1 will be (n − 2)! and put it after a_1 in those sub-grooves. Similarly, with a_4, a_5,...., a_n. Again amongst the sub-grooves that can have any other particular thing in the third place will be (n − 3)! and it should be placed in those cases. Proceeding step by step in this way in a systematic manner, we can find out all the different permutations of things.

Piṅgala's Rules

Piṅgala (before 200 BC) describes a scheme of forming all the permutations with a specified number of things when repetitions are allowed. As he was directly concerned with metres, he dealt with only two varieties of things, long and short syllables, which are represented respectively by the abbreviations *g* from *guru* ("long") and *l* from *laghu* ("short"). But the scheme is equally applicable to cases of more varieties. Piṅgala's scheme, described in short aphorisms[47], will be clear from the following:

(i)	Monosyllabic:	1. *g*
		2. *l*

(ii) Disyllabic:

$$\left.\begin{array}{l}\left.\begin{array}{l}g\\ l\end{array}\right\}g\\ \left.\begin{array}{l}g\\ l\end{array}\right\}l\end{array}\right\} = \left\{\begin{array}{ll}1. & gg\\ 2. & lg\\ 3. & gl\\ 4. & ll\end{array}\right.$$

(iii) Trisyllabic:

$$\left.\begin{array}{l}\left.\begin{array}{l}gg\\ lg\\ gl\\ ll\end{array}\right\}g\\ \left.\begin{array}{l}gg\\ lg\\ gl\\ ll\end{array}\right\}l\end{array}\right\} = \left\{\begin{array}{ll}1. & ggg\\ 2. & lgg\\ 3. & glg\\ 4. & llg\\ 5. & ggl\\ 6. & lgl\\ 7. & gll\\ 8. & lll\end{array}\right.$$

and so on. Piṅgala states that the trisyllabics are 8 in number[48]. In general, a group of n syllables will have 2^n forms (*vide infra*).

The above systematic scheme of representation has the advantages that (*a*) we can easily find out the form of versification corresponding to a given serial number in it and vice versa, (*b*) we can allocate a given form of versification in its proper place in the scheme. Piṅgala's aphorisms for (*a*) are, "*l* when halved; *g* when added with one (and then halved)"[49]. That is to say: Divide the given number successively by two; if at any step, the number obtained is not divisible by two, add one to it and then halve. Corresponding to each operation of exact division by two, set down *l*; and to that of halving after adding unity write down *g*. The operations are to be continued until the desired number of syllables in the group has been obtained. The operations for (*b*) are the reverse of these [50]. Taking unity, we shall have to double it successively as many times as there are syllables in the given form; but corresponding to each long syllable we shall have to subtract one from the corresponding product.

Piṅgala next gives a rule for finding the total number of variations without having recourse to writing them all down exhaustively according to the scheme described above. This method has already been described. It is found in later writings also[51]. By this rule, the total number of variations in a group of n syllables is found to be equal to 2^n.

Piṅgala has also an alternative method to find the total number of variations[52]. It is technically called *Meru-prastāra*, because the total is obtained by addition from numbers arranged in such a form as to present a fancied resemblance to the fabulous mountain Meru of the Hindu mythology. Piṅgala's aphorisms being too compressed and cryptic can be understood only with the help of a commentary. Halāyudha (10th century) has explained them as follows:

"Draw one square at the top; below it draw two squares, so that half of each of them lies beyond the former on either side of it. Below them in the same way draw three squares; then below them four; and so on up to as many rows as desired: this is the preliminary representation of the Meru (*Meru-prastāra*). Then putting down one in the first square, the marking should be started. In the next two squares write one in each. In the third row, put 1 in each of the two extreme squares and in the middle square, the sum of the two digits in the two squares of the second row. In the fourth row, put 1 in the two extreme squares; in an intermediate square put the sum of the digits in two squares of the previous row, which lie just above it. Putting down numbers in the other rows should be carried on in the same way. Now the numbers in the second row of squares show the monosyllabic forms: There are two forms, one consisting of a long and the other of a short syllable. The numbers in the third row give the disyllabic forms: in one form all syllables are long; in two forms one syllable is short; and in one all syllables are short. In this row of the squares we get the number of variations of the even verse. The numbers in the fourth row of squares represent trisyllabics. There one form has all syllables long, three have one short syllable; three have two short syllables and one has all syllables short, and so on. In the fifth and succeeding rows also the figure in the first square gives the number of forms with all syllables long, that in the last all syllables short and the figures in the successive intermediate squares represent the number of forms with one, two, etc. short syllables"[53].

Thus, according to the above, the number of variations of a metre containing n syllables will be obtained from the representation of the *Meru* as follows:

Number of syllables		Total number of variations	
	1		
1	1 1	2	2^1
2	1 2 1	4	2^2
3	1 3 3 1	8	2^3
4	1 4 6 4 1	16	2^4
5	1 5 10 10 5 1	32	2^5
6	1 6 15 20 15 6 1	64	2^6

From the above it is clear that Piṅgala knew the results:

(1) ${}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1} + {}^nC_{n+1} = 2^n$,

(2) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

Sanskrit prosody distinguishes three classes of metres: (1) even, in which the arrangement of syllables in all the quarters (*padas*) is the same; (2) semi-even, in which the alternate quarters are alike; and (3) uneven, in which the quarters are all dissimilar. Now with a group of n syllables in a quarter, the total number of varieties of even metres will be, according to Piṅgala[54], 2^n; semi-even, $2^{2n}-2^n$; and uneven, $2^{4n}-2^{2n}$. The same formulae are stated also by Bhāskara II:

"The number of syllables in a quarter being taken for the period and the common ratio 2 the result from multiplication and squaring[55] will give the number of even metres. Its square, and square's square, minus their respective roots, will be the numbers of semi-even and uneven metres respectively"[56].

By way of illustration, Halāyudha[57] calculates that in the *Gāyatrī* metre, which has six syllables in a quarter, the number of even variations will be 64, semi-even 4032, and uneven 16773120. Bhāskara II[58] calculates that in the case of the *Anuṣṭubh* metre, which has 8 syllables in a quarter even variations are 256, semi-even 65280, and uneven 4294901760.

Nemicandra's Rules

We find in the works of Nemicandra, a Jaina philosophical writer of the tenth century (*c.* 975), certain interesting rules, some of which are akin to those of Piṅgala. According to the Jaina philosophy, there are 15 kinds of *pramāda* ("carelessness"), of which four belong to the category of *vikathā* ("wrong talk"), four to that of *kaṣāya* ("passion"), five to that of *indriya* ("sense") and one each to those of *nidrā* ("sleep") and *praṇaya* ("attachment"). Combinations are made of five elements of carelessness, selecting only one element from each of the five categories. Again, they are formed by setting down the elements according to a systematic scheme and are marked serially. Hence, the problems that arise in this connection are, as enumerated by Nemicandra, to find: (1) the total number of combinations that can be made, (ii) a systematic scheme of laying out, (iii) the elements of a combination from its serial number, and (iv) the serial number of a particular combination[59]. Nemicandra has given rules for each.

(i) "All the combinations previously obtained combine with each element of the next category. Hence, the total number will be given by the multiplication (of the numbers of elements in the different categories)"[60].

Thus, the total number of combinations that can be made out of the 15 elements of carelessness in the way described above is $4\times4\times5\times1\times1=80$.

(ii) Nemicandra has described two schemes of representation of combinations: one is called (1) the *prastāra* and the other (2) the *parivartana*.

(1) "The distribution will be obtained thus: Write down severally the first element (of the first category) of carelessness and put over it each of the elements of the succeeding classes. When elements in the third category are exhausted, begin afresh with the second element of the second category (and so on). When all the elements of these two categories are thus distributed out, operations should be begun with (the second element of) the first category. (And so on)"[61].

If the four kinds of wrong talks be denoted by v_1, v_2, v_3, v_4; the four kinds of passions by k_1, k_2, k_3, k_4; and the five kinds of senses by i_1, i_2, i_3, i_4, i_5, the representation described here will be this:

$$\begin{array}{c} \underbrace{i_1\ i_2\ i_3\ i_4\ i_5} \\ k_1 \\ v_1 \end{array} \qquad \begin{array}{c} \underbrace{i_1\ i_2\ i_3\ i_4\ i_5} \\ k_2 \\ v_1 \end{array} \qquad \begin{array}{c} \underbrace{i_1\ i_2\ i_3\ i_4\ i_5} \\ k_3 \\ v_1 \end{array}$$

$$\begin{array}{c} \underbrace{i_1\ i_2\ i_3\ i_4\ i_5} \\ k_4 \\ v_1 \end{array}$$

and so on, v_2, v_3, v_4 coming successively in the places of v_1.

(2) "Write down the elements of the first category as many times as the number of elements in the second category; then put down over each group severally each of the elements of the second category; and proceed thus throughout. When all the elements of the first category are exhausted, begin afresh with the second category (and so on). When all the elements of these two categories are thus distributed out, the operation with the elements in the third category begins"[62].

$$\begin{array}{c} i_1 \\ k_1 \\ \overbrace{v_1\ v_2\ v_3\ v_4} \end{array} \qquad \begin{array}{c} i_1 \\ k_2 \\ \overbrace{v_1\ v_2\ v_3\ v_4} \end{array} \qquad \begin{array}{c} i_1 \\ k_3 \\ \overbrace{v_1\ v_2\ v_3\ v_4} \end{array} \qquad \begin{array}{c} i_1 \\ k_4 \\ \overbrace{v_1\ v_2\ v_3\ v_4} \end{array}$$

(iii) "Divide (the given serial number) successively by the number of elements in the different categories, adding each time unity to the quotient, except when the remainder is zero. The remainders determine the place of an element in its category; the zero remainder indicates the last element"[63].

For example, let us find the elements of the 13th combination.

First Scheme: Dividing 13 by 5, we get 2 for the quotient and 3 as the remainder. So, the combination contains the element i_3. Adding 1 to the quotient 2, we have 3. Dividing 3 by 4, we get the quotient 0 and the remainder 3. Hence, there is the element k_3, $0+1=1$. On dividing 1 by 4, the remainder is 1. So, there is v_1. So, the 13th combination according to the first scheme contains i_3, k_3 and v_1, besides sleep and attachment.

Second scheme: On dividing 13 by 4, the quotient is 3 and the remainder 1. So, the combination contains v_1. Adding unity to the quotient 3, we get 4. On dividing 4 by 4, the quotient is 1 and the remainder 0. Hence, there is k_4[64]. Dividing 1 by 5, we get the remainder 1. So, there is i_1. Hence, the 13th combination according to the second scheme has v_1, k_4, i_1, besides sleep and attachment.

(iv) "Take unity. Multiply it by the total number of elements in a category beginning from the last and subtract from the product the number of elements there following the given element. Proceed in the same way throughout"[65].

For example, let us find the number of the combination $i_4k_3v_1$. Take 1. As there are 4 elements in the last category v, we multiply it by 4 and get $1 \times 4 = 4$. Since there are only 3 elements in that category after v_1, we subtract 3 from the product and get $4 - 3 = 1$. Next, we shall have to multiply the remainder 1 by 4, since there are 4 elements in the category of k and subtract from the result 1, since there lies only 1 element in the category after k_3. Thus, we get $1 \times 4 - 1 = 3$. Now we multiply 3 by 5, there being 5 elements in the category of i and then subtract 1, there being only one element after i_4. So, we get $3 \times 5 - 1 = 14$. Hence, the serial number of the combination $i_4k_3v_1$ is 14.

To get the same results as stipulated in rules (iii) and (iv) more easily and quickly, without going through the lengthy process of calculations described therein, Nemicandra gives two short tables. He says:

Table 1

"Place 1, 2, 3, 4, 5; 0, 5, 10, 15; 0, 20, 40 and 60 in three rows (of cells) of the three categories of carelessness, and find the elements and the serial numbers of combinations"[66].

i_1 1	i_2 2	i_3 3	i_4 4	i_5 5
k_1 0	k_2 5	k_3 10	k_4 15	
v_1 0	v_2 20	v_3 40	v_4 60	

Table 2

"Set down 1, 2, 3, 4; 0, 4, 8, 12; 0, 16, 32, 48 and 64 in three rows (of cells) of the three categories of carelessness, and find the elements and the serial numbers of combinations"[67].

v_1 1	v_2 2	v_3 3	v_4 4	
k_1 0	k_2 4	k_3 8	k_4 12	
i_1 0	i_2 16	i_3 32	i_4 48	i_5 64

Table 1 is to be used in case of distribution on the first scheme and Table 2 in that on the second scheme. To find the serial number of a given combination, we have simply to add together the figures placed in the cells of its elements in the tables. And to determine the elements occurring in a combination whose serial number is given, we shall have to break up that number into three parts picked up from three rows of cells in the tables and then write down in order the elements from those cells.

For example, since $13 = 3 + 10 + 0$, the 13th combination in the first scheme will be $i_3k_3v_1$ as determined before. According to the second scheme, it will be $v_1k_4i_1$, since $13 = 1 + 12 + 0$.

Notes and References

1. See the article of Gurugovinda Chakravarti on the "Growth and development of permutations and combinations in India" in *BCMS*, XXIV (1932).
2. See Ch. xiv, vv. 55-81.
3. See Ch. lxiii.
4. For instance see *Jambūdvīpa-prajñapti* xx. 4, 5; *Bhagavatī-sūtra, Sūtras* 314, 341, 371-4, etc.; *Anuyogadvāra-sūtra, Sūtras* 76, 92, 126. Compare Bibhutibhusan Datta, "The Jaina School of Mathematics", *BCMS*, XXI (1929), pp. 133 ff.
5. See *L* (= *Līlāvatī*) p. 26 f. *Cf. GK* (= *Gaṇita-Kaumudī*), xiii. 2.
6. See *BrSpSi* (= *Brāhma-sphuṭa-siddhānta*), xx.
7. See *GSS* (= *Gaṇita-sāra-saṅgraha*), Ch. vi.
8. See *PG* (= *Pāṭīganita*), p. 95.
9. See *L*, p. 83.
10. See *GK*, II, p. 286.
11. In *Suśruta-saṁhitā*, Ch. lxiii.
12. *Bhagavatī-sūtra, Sūtra* 314.
13. *Bṛhat-saṁhitā*, with the commentary of Bhaṭṭotpala, edited by Sudhākara Dvivedī, in two volumes, Benaras, 1897, lxxvi, 22.
14. *Bṛhat-saṁhitā*, lxxvi. 13-21. See also lxxvi. 29-30.

    It will be noted that the total number of perfumes will be 24×1820, i.e. 43680, and not 174720, as stated by Varāhamihira. His commentator Bhaṭṭotpala rightly remarks: "This number (i.e. 174720) is obtained by taking all varieties subordinate to each ingredient, and not by taking the main varieties (which must be all different). Considering the main varieties only, the total number of perfumes comes to 43680, because a group of four yields only 24 varieties (of perfumes)".
15. *Bṛhajjātaka*, edited by Sitarama Jha, with the commentary of Bhaṭṭotpala, Benaras, 1921, Ch. xiii, vs. 4.
16. *PG*, Rule 72.
17. *GSS*, vi. 218.
18. PG, Ex. 95, *GSS*, vi. 19.
19. *L*, p. 27.
20. *GK*, xiii. Ex. 22.
21. *Bṛhajjātaka*, xii. 19(com).
22. *L*, p. 27.
23. *GK*, xiii. 59.
24. *Anuyogadvāra-sūtra, Sūtra* 97.
25. *Ibid, Sūtras* 103, 114-9.
26. *Viśeṣāvaśyaka-bhāṣya, Gāthā* 942.
27. *Vide* Śīlāṅka's comm.on *Sūtrakṛtāṅga sūtra, samayādhyayana, anuyògadvāra*, verse 28.
28. *L*, p. 83.
29. *L*, p. 84.
30. *GK*, xiii. 45, 91.

31. *L*, p. 84.
32. *GK*, xiii. 55(c – d)-56(a – b).
33. *L*, p. 84.
34. *GK*, xiii. 62(a).
35. *GK*, xiii. Ex. 27.
36. *L*, p.83.
37. *L*, p. 84.
38. *L*, pp. 83, 84.
39. *Cf.* Haran Chandra Banerjee; *Līlāvatī*, Second edition, Calcutta (1927), pp. 192-195.
40. *L*, p.85.
41. *L*, p. 85.
42. *Bhagavatī-sūtra, Sūtra* 314.
43. *Bṛhat-saṁhitā*, lxxvi. 22; *Bṛhajjātaka*, xiii. 4.
44. Vide Bhaṭṭotpala's commentary on *Bṛhajjātaka*, xii. 19.
45. *Viśeṣāvaśyaka-bhāṣya*.
46. loc. cit; *Cf.* B. Datta, *Jaina Math*, pp. 135 f.
47. *Chandaḥ-sūtra of Piṅgala*, edited by Jīvananda Vidyasagara, with the commentary of Halāyudha, Calcutta, 1892; viii. 20-2.
48. *Chandaḥ-sūtra of Piṅgala*, viii. 23.
49. *Ibid*, viii. 24-5.
50. *Ibid*, viii. 26-7.
51. For instance, Mahāvīra (*GSS*, ii. 94). Pṛthūdakasvāmī (BrSpSi, xii. 17. com.), etc.
52. *Chandaḥ-sūtra*, viii. 28-32.
53. *Chandaḥ-sūtra*, viii. 33-4.
54. *Chandaḥ-sūtra*, v. 3-5.
55. Reference here is to the operations described for finding the sum of a G.P. (*L*, p. 31).
56. *L*, p. 31.
57. See his commentary on *Chandaḥ-sūtra*, v. 3-5.
58. *L*, p. 32.
59. *Gommaṭasāra, Jīvakāṇḍa, Gāthā* 35.
60. *Ibid, Gāthā* 36.
61. *Gommaṭasāra, Jīvakāṇḍa, Gāthās* 37, 39.
62. *Ibid, Gāthās* 38, 40.
63. *Ibid, Gāthā* 41.
64. As there is no element with zero suffix, the remainder gives k_4.
65. *Ibid, Gāthā* 42.
66. *Ibid, Gāthā* 43.
67. *Ibid, Gāthā* 44.