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MAGIC SQUARES IN INDIA

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A square containing an equal number of cells in each row and each column is called a magic square, when the total of numbers in the cells of each row, each column and each diagonal happens to be the same. Magic squares have been known in India from very early times. It it believed that the subject of magic squares was first taught by Lord Śiva to the magician Maṇibhadra. Magic squares are said to have magical properties and were used in various ways by the Hindus as well as the Jainas. But the mathematics involved in the construction of magic squares and other magic figures was first systematically and elaborately discussed by the mathematician Nārāyaṇa (AD 1356) in his *Gaṇitakaumudī*. Some of his methods were unknown in the west and were recently discovered by the efforts of several scholars. The present article, besides giving a brief history of magic squares, explains the methods given by Nārāyaṇa and other Hindu writers for the construction of magic squares of various types

ORIGIN AND EARLY HISTORY

Little is known as regards the origin of magic squares and other figures. Hindu tradition assigns them to God Śiva. Nārāyaṇa (1356) says that the subject of progression, of which magic squares form a part, was taught by Śiva to Maṇibhadra*, the magician. The earliest unequivocal occurrence of magic squares is found in a work called *Kakṣapuṭa* composed by the celebrated alchemist and philosopher Nāgārjuna who flourished about the 1st century AD. One of the squares in this work is called *Nāgārjunīya* after him; so there can be no doubt that he did really construct some squares. The squares given by Nāgārjuna are all 4 ×4 squares, and some of these seem to have been known before him. The easier case of 3×3 square must have also been known earlier to Nāgārjuna. Another square is found in a work of Varāhamihira (d. 587 AD).

4×4 magic squares are considered to possess magical properties and are supposed to bring luck when worn as amulets. They are found on gates of buildings, on the walls where shopkeepers transact their business and on the covers of calendars used by astrologers even to this day. A 4×4 square occurs in a Jaina inscription of the 11th century, found in the ancient town of Khajuraho.

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*Reference to Maṇibhadra Yakṣa occurs in the Buddhist work *Saṃyukta-nikāya* (i.10,4) and the Jaina work *Sūrya-prajñapti*. See D.N. Shukla, *Pratimā-Vijñāna* (in Hindi), p. 51.

A systematic study of magic figures was taken up by Nārāyaṇa, who in his *Gaṇita Kaumudī* (1356) gives general methods for the construction of all sorts of magic squares with the principles governing such constructions. He seems to have been the first to conceive of other figures in which numbers may be arranged so as to possess properties similar to those of magic squares. An account of the methods of constructing magic squares given by the authors mentioned above and also by other Hindu writers is given in this article.

It is the opinion of some historians of mathematics that magic squares first originated in China. This opinion is based on the occurrence of a square, filled with white and black dots, in the introduction of a Chinese work, the *I-king*. The square is called the *Loh Shu*, and is said to have come down to us from the time of the great emperor Yu (*c.* 2200 BC). According to Chinese tradition, the white dots denote odd numbers and the black dots even ones, and it has been conjectured that the *Loh Shu* represents the square

4	9	2
3	5	7
8	1	6

Fig. 1

But to consider the *Loh Shu* as a magic square is to force upon it an interpretation which it originally did not possess. Arrangements of white and black dots in the figure of a square are met with elsewhere in the literature of the Chinese. One such arrangement represents the river *Ho* and has nothing to do with magic squares[1].

The first unequivocal appearance of the *Loh Shu* in the form of a magic square is found in the writings of Tsai Yuan-Ting[2] who lived from 1135 to 1198 AD. Magic squares occur also in the writings of Hebrew[3] and Arab[4] scholars about the same period, while in India they were used much earlier. It would thus appear that the Chinese claim to the invention of magic squares is not well founded[5].

Nāgārjuna Squares

In his *Kakṣapuṭa*, Nāgārjuna (100) gives rules for the construction of 4×4 squares with even as well as odd totals[6]. These rules consist partly of mnemonic verses in which numbers are expressed in alphabetic notations. The general direction is

Arka	*Indunidhā*	*Nārī*	*tena*	*lagna*	*viuāsanam*
0 1	0 8 0 9	0 2	6 0	3 0	4 0 7 0

By inserting these values in the successive cells (of the 4×4 square) leaving blanks for zero, we get the primary skeleton

	1		8
	9		2
6		3	
4		7	

Fig. 2

The eight blank cells can be filled up in such a way as to give even as well as odd totals. But the methods of filling up differ slightly in the two cases.

(i) *Even Total:* In order to have an even total, fill up, says Nāgārjuna, the blank cells by writing the difference between half of that total and the number in the alternate cell in a diagonal direction from the cell to be filled up. This direction may be upwards or downwards, right or left.

Taking the total to be 2n, where n is any integer, we thus get the complete magic square with even totals.

n−3	1	n−6	8
n−7	9	n−4	2
6	n−8	3	n−1
4	n−2	7	n−9

Total = 2n

Fig. 3

In this magic square, the totals of all the rows, horizontal, vertical, and diagonal, of every group of four forming a sub-square, and separated by such a sub-square, and of the corner four of the square and about a small square, are equal. Another noteworthy feature of it is that each of its four minor squares has relation to others, as may be seen in Fig. 4.

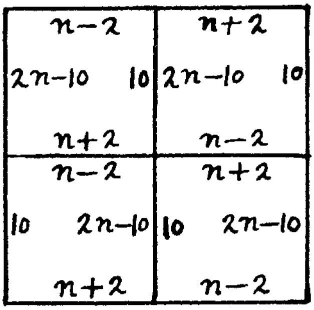


Fig. 4

The above square is "continuous" according to the definition of Paul Carus; that is, "It may vertically as well as horizontally be turned upon itself and the rule holds good that wherever we may start four consecutive numbers in whatever direction, backward or forward, upward or downward, in horizontal, vertical or slanting lines, always yield the same sum....and so does any small square of 2×2 cells"[7]. Since the square cannnot be bent upon itself at once in two directions, the result is shown in Fig. 5, by extending the square in each direction by half its own size.

3	n−1	6	n−8	3	n−1	6	n−8
7	n−9	4	n−2	7	n−9	4	n−2
n−6	8	n−3	1	n−6	8	n−3	1
n−4	2	n−7	9	n−4	2	n−7	9
3	n−1	6	n−8	3	n−1	6	n−8
7	n−9	4	n−2	7	n−9	4	n−2
n−6	8	n−3	1	n−6	8	n−3	1
n−4	2	n−7	9	n−4	2	n−7	9

(ii) *Odd Total:* For an odd total, say 2n + 1, we are to fill up the blank cells by writing the difference between n and the number in the alternate cell in a diagonal direction from the cell to be filled up, when the latter number happens to be 1, 2, 3 or 4; or the difference between n + 1 and the number in the alternate cell in a diagonal direction from the cell to be filled up, if the latter number be 6, 7, 8 or 9. This direction may be, as in the previous case, upwards or downwards, right or left. Proceeding in this way, we get the complete magic squares having an odd total.

n−3	1	n−5	8
n−6	9	n−4	2
6	n−7	3	n−1
4	n−2	7	n−8

Total = 2n + 1

Fig. 6

In this case , the totals of all rows, horizontal, vertical and diagonal, of every group of four forming a square (except the group of the fifth, sixth, ninth and tenth cells, and that of the seventh, eighth, eleventh and twelfth cells), of the corner four of the square, and of the four about the corners of a small square are equal. The relation between the four minor squares in this case is not as complete as in the previous case.

n−2 2n−9 10 n+3	n+3 2n−9 10 n−2
n−1 10 2n−9 n+2	n+2 10 2n−9 n−1

Fig. 7

It is not a perfectly continuous square. The odd totals cannot be less than 19 in any case, and not less than 37 if the same number is not to appear more than once in the square. (See Fig.6)

A particular case of 4×4 squares with even total, 100, has been specially noted by Nāgārjuna. Its form differs from that which results on putting n = 50 in the above general case, and further it does not contain the numbers from 1 to 9, except 6.

30	16	18	36
10	44	22	24
32	14	20	34
28	26	40	6

Total = 100
Fig. 8

This magic square has been called the Nāgārjunīya[8]. This special epithet will lead one to presume that this particular square was constructed by Nāgārjuna, while others described by him were recapitulations of former accomplishments.

Varahamihira Square

Varāhamihira (d. 587 AD) gives a form of 4×4 magic squares, viz.[9]

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

Total = 18
Fig. 9

in which the total is 18. It is, however, a particular case of the following:

n−7	3	n−4	8
5	n−1	2	n−6
4	n−8	7	n−3
n−2	6	n−5	1

Total = 2n

Fig. 10

2	n−6	5	n−1
n−4	8	n−7	3
n−5	1	n−2	6
7	n−3	4	n−8

Total = 2n

Fig. 11

Varāhamihira has called his square *Sarvatobhadra* ("Magic in all respects") and what are implied by that name, i.e., the special features of the square, have been pointed out fully by his commentator, Bhaṭṭotpala (966). Indeed it has properties similar to those of squares with even totals described by Nāgārjuna. The method of filling up the blank cells in the primary skeleton is the same.

	3		8
5		2	
4		7	
	6		1

Fig. 12

2		5	
	8		3
	1		6
7		4	

Fig. 13

The blank cells can be filled up so as to yield also an odd total; write the difference between n + 1 and the number in the alternate cell in a diagonal direction from the cell to be filled up, if the latter number happens to be 1, 2, 3 or

4, or the difference between n and the number in the alternate cell in a diagonal direction from the cell to be filled up, if the latter number happens to be 5, 6, 7 or 8.

$n-7$	3	$n-3$	8
5	n	2	$n-6$
4	$n-8$	7	$n-2$
$n-1$	6	$n-5$	1

Total = 2n + 1

Fig. 14

2	$n-6$	5	n
$n-3$	8	$n-7$	3
$n-5$	1	$n-1$	6
7	$n-2$	4	$n-8$

Total = 2n + 1

Fig. 15

JAINA SQUARES

In a Jaina inscription found amongst the ruins of the ancient town of Khajuraho occurs a magic square of 4 x4 cells of which the total is 34 (Fig. 16). It possesses all the special features of the Nāgārjuna squares. It belongs to the eleventh century of the Christian era. In the *Tijapapahutta Stotra* of the Jainas, we find another 4×4 magic square having a total of 170[10] (Fig. 17).

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Total = 34

Fig. 16

25	80	15	50
20	45	30	75
70	35	60	5
55	10	65	40

Total = 170

Fig. 17

The date of this square is uncertain. It is certainly not later than the fourteenth century, when a commentary on the above *Stotra* (hymn) was written. It is probably a very old one. Its total 170 is closely connected with an ancient Jaina mythology about the appearance of their prophets.

Nārāyaṇa's Results

As has been already pointed out, the only Hindu work, known to us, which gives a systematic mathematical treatment of the construction of magic squares and other figures is the *Gaṇita Kaumudī* of Nārāyaṇa. Chapter XIV of the work is devoted to this subject, and we propose to give here a summarized version of this chapter, preserving the order of treatment and giving explanatory notes wherever necessary.

Summary

In order to bring into prominence the remarkable achievements of Nārāyaṇa in the theory of magic squares, it is thought desirable to state briefly some of his most important results before entering into details. These results are:

1. Magic squares are of three types: (a) those which have 4n cells in a row, (b) those which have $4n+2$ cells in a row, and (c) those which have an odd number of cells in a row.

2. Series in arithmetical progression are used for the construction of these squares.

3. Magic squares can be made of as many series or groups of numbers as there are cells in a column.

4. Each series or group is composed of as many numbers as there are groups.

5. The common difference must be the same for each group.

6. The initial terms of the groups are themselves in A.P.

7. The numbers is a group, although belonging to an arithmetical progression, may be disarranged in various ways for the filling of the square.

8. The method of the knight's move for the construction of a $4n \times 4n$ square.

9. The method of superposition for the construction of $4n \times 4n$ squares.

10. The method of equi-spacing for the construction of $(4n+2) \times (4n+2)$ squares.

11. The method of superposition for odd squares.

12. A special method for odd squares.

13. The construction of a magic rectangle (*vitāna* or canopy).

14. The construction of magic circles, triangles, hexagons and various other figures, such as the altar, the diamond, etc.

Preliminary Remarks

According to Nārāyaṇa, magic squares may be classified into three groups: (1) *samagarbha*, (2) *viṣamagarbha*, and (3) *viṣama*. These terms are defined as follows:

> "If on dividing the *bhadrāṅka*[11] ("number of cells in a line of the square") by four, the remainder is zero, the magic square is said to be *samagarbha*. If the remainder is two, it is called *viṣamagarbhā*; and if the remainder is one or three, it is simply *viṣama*[12].

After giving the above classification, Nārāyaṇa remarks:

> "In the construction of magic squares, the arithmetical progression is used[13]. In relation to that (magic square) which is required to be constructed, first find the *initial term* and the *common-difference* (of a series in arithmetical progression, corresponding to the given sum and the number of cells)[14]. The sum divided by the *bhadrāṅka* ("number of the square") gives the *phala* ("total"). The number of terms to be taken in the progression is the number of *gṛha* ("cells") in the square[15]. If the number of cells *(koṣṭha)* is a square number, its root is called the *carana* ("foot" or "row"). Such are the technical terms used by Nārāyaṇa in his *bhadragaṇita* ("calculations relating to magic figures")[16].

The method of finding out the initial term and the common-difference of an arithmetical progression, given the sum and the number of terms, follows the above preliminary remarks[17].

The 4 × 4 Magic Squares[18]

Assuming that the required arithmetical progression has been found out, Nārāyaṇa gives the following rule for filling the cells of the 4 × 4 square with the numbers occurring in the progression:

> "In the manner of the chess-board, place the numbers forming the progression, (taking them) two and two, in two connected cells as well as in alternate cells, in the direct and inverse order. (Then) by right and left knight's move fill the cells (of the square) with the numbers (taking them as they have been placed above). This method has also been stated by previous teachers for the

construction of the *samagarbha* magic square of sixteen cells. The numbers, in the horizontal cells, in the vertical cells as well as in the diagonal cells, added separately, give rise to the same total"[19].

Example from Nārāyaṇa:

"Friend, tell me, how a 4×4 magic square be filled up with the numbers beginning with unity and successively increasing by one, so that the horizontal, vertical and diagonal cells shall have the same sum".

Here the sum of the natural numbers from one to sixteen is $(16 \times 17)/2 = 136$. Therefore, the required total is $136/4 = 34$.

The numbers when written, two and two, in connected cells as well as alternate cells give Figs. 18 and 19.

(a_0)

1	2	3	4
8	7	6	5
9	10	11	12
16	15	14	13

Fig. 18

(b_0)

1	3	2	4
8	6	7	5
9	11	10	12
16	14	15	13

Fig. 19

Placing two and two in the direct and inverse orders, we get Figs. 20 and 21.

(a)

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

Fig. 20

(b)

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

Fig. 21

[Note that the numbers in the 4 cells on the right of the first two rows are reversed and the same is done with the numbers in the 4 cells on the left of the last two rows. This would probably be an easier method of stating the method.]

Taking the arrangement (a), and filling by right and left knight's move, the four squares depicted in Figs. 22-25 are obtained.

(a_1)

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Fig. 22

(a_2)

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

Fig. 23

(a_3)

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

Fig. 24

(a_4)

1	14	4	15
12	7	9	6
13	2	16	3
8	11	5	10

Fig. 25

The arrangement (b) similary gives the four squares as per Figs. 26-29.

(b_1)

1	8	13	12
15	10	3	6
4	5	16	9
14	11	2	7

Fig. 26

(b_2)

1	15	4	14
8	10	5	11
13	3	16	2
12	6	9	7

Fig. 27

(b_3)

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

Fig. 28

(b_4)

1	15	4	14
12	6	9	7
13	3	16	2
8	10	5	11

Fig. 29

The filling in of the numbers begins by putting 1 in the first cell. After the first row of numbers is exhausted, we begin by putting the first number of the second row (i.e., 8) in a contiguous cell, as in (a_1) or (a_2). [In order to make up the desired total, 10 is made to correspond to 8 and 15 to 1, as in the illustration above.] If the square be considered to be wrapped round a cylinder, the fourth cell is also contiguous to the first cell; hence the squares (a_3) and (a_4). It may be noted that (a_2) may be obtained by turning (a_1) through a right angle, whilst (a_3) and (a_4) may be obtained by wrapping (a_1) and (a_2) round a cylinder.

VARIETIES OF 4×4 MAGIC SQUARES

Nārāyaṇa remarks:

"Here, other 4×4 squares may be produced from a 4×4 square by turning four cells to make the numbers inverse"[20].

"In the rest of the *caraṇa* ("row") following the first cell, (by turning) four numbers produced in two connected pairs of cells, there result twenty-four varieties. And the same numbers arise from others separately"[21].

Example from Nārāyaṇa:

How many 4×4 squares can be formed out of the series of natural numbers from one to sixteen, and what are their forms?[22]

The numbers are placed according to the previous rule as per Figs. 30 and 31.

(a)

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

Fig. 30

(b)

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

Fig. 31

By turning four cells, i.e., the two connected pairs in the middle of the first two rows, and doing the same for the last two rows, we have Figs. 32 and 33

(a')

1	5	7	3
8	4	2	6
10	14	16	12
15	11	9	13

Fig. 32

(b')

1	5	6	2
8	4	3	7
11	15	16	12
14	10	9	13

Fig. 33

Performing the same operation on the two connected pairs of cells at the end in (a) and (b), we have Figs. 34 and 35.

(a″)

1	2	6	5
8	7	3	4
10	9	13	14
15	16	12	11

Fig. 34

(b″)

1	3	7	5
8	6	2	4
11	9	13	15
14	16	12	10

Fig. 35

The numbers in the arrangements (a′), (b′), (a″) and (b″) are filled in the 4×4 square in the same way as those of (a) or (b). Thus, there will be altogether 24 squares with 1 in the first cell. As there are sixteen numbers, so there can be 384 varieties of 4×4 squares, formed out of the series of natural numbers one to sixteen.

The twenty-four varieties with 1 in the first cell have been shown by Nārāyaṇa as per Fig. 36.

Example: In a certain 4×4 square, the total *(phala)* is 40, find the initial term and the common-difference. Also find them when the total is 64.

The equations giving the initial term (a) and the common difference (d) are:

(i) $10 - \frac{15}{2} d = a$, when the total is 40;

and

(ii) $16 - \frac{15}{2} d = a$, when the total is 64.

These give: for case (i) $a = -5, \ldots; d = 2, \ldots$

and for case (ii) $a = 1, -14, \ldots; d = 2, 4, \ldots$

NĀRĀYAṆA'S SQUARES

[1]

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

[5]

1	8	13	12
15	10	3	6
4	5	16	9
14	11	2	7

[9]

1	8	10	15
14	11	5	4
7	2	16	9
12	13	3	6

[2]

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

[6]

1	15	4	14
8	10	5	11
13	3	16	2
12	6	9	7

[10]

1	14	7	12
8	11	2	13
10	5	16	3
15	4	9	6

[3]

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

[7]

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

[11]

1	15	10	8
14	4	5	11
7	9	16	2
12	6	3	13

[4]

1	14	4	15
12	7	9	6
13	2	16	3
8	11	5	10

[8]

1	15	4	14
12	6	9	7
13	3	16	2
8	10	5	11

[12]

1	14	7	12
15	4	9	6
10	5	16	3
8	11	2	13

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

1	5	7	3
8	4	2	6
13	9	11	15
12	16	14	10

[13]

1	8	11	14
15	10	5	4
6	3	16	9
12	13	2	7

[17]

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

[21]

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

[14]

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

[18]

1	12	6	15
8	13	3	10
11	2	16	5
14	7	9	4

[22]

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

[15]

1	14	11	8
15	4	5	10
6	9	16	3
12	7	2	13

[19]

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

[23]

1	15	10	8
12	6	3	13
7	9	16	2
14	4	5	11

[16]

1	15	6	12
14	4	9	7
11	5	16	2
8	10	3	13

[20]

1	12	6	15
14	7	9	4
11	2	16	5
8	13	3	10

[24]

1	12	7	14
15	6	9	4
10	3	16	5
8	13	2	11

1	5	6	2
8	4	3	7
13	9	10	14
12	16	15	11

1	2	6	5
8	7	3	4
10	9	13	14
15	16	12	11

1	3	7	5
8	6	2	4
11	9	13	15
14	16	12	10

Fig. 36.

The squares constructed, according to the rule given above, with the above values of a and d are shown in Figs. 37-39.

−5	9	19	17
21	15	−3	7
1	3	25	11
23	13	−1	5

Total = 40

Fig. 37

1	15	25	23
27	21	3	13
7	9	31	17
29	19	5	11

Total = 64

Fig. 38

−14	14	34	30
38	26	−10	10
−2	2	46	18
42	22	−6	6

Total = 64

Fig. 39

Use of Irregular Series

Instead of employing 16 numbers in arithmetical progression to fill up a 4×4 square, four different arithmetic series, with different initial terms but the same common-difference consisting of four terms each may be used[23]. Nārāyaṇa gives the following examples to illustrate this:

Examples from Nārāyaṇa[24].

(a) To construct 4 × 4 magic squares with total 40.

In this case, the *caraṇas* (rows, i.e., the arithmetic progressions of which each has as many terms as there are cells in a row) may be supposed to be:

(i)	1	2	3	4	(ii)	1	2	3	4	(iii)	2	3	4	5
	6	7	8	9		5	6	7	8		6	7	8	9
	11	12	13	14		12	13	14	15		11	12	13	14
	16	17	18	19		16	17	18	19		15	16	17	18

Now filling up the cells by the same method as before, we get Figs. 40-42.

1	9	16	14
17	13	2	8
4	6	19	11
18	12	3	7

Total = 40

Fig. 40

1	8	16	15
17	14	2	7
4	5	19	12
18	13	3	6

Total = 40

Fig. 41

2	9	15	14
16	13	3	8
5	6	18	11
17	12	4	7

Total = 40

Fig. 42

(b) To construct 4 × 4 squares with total 64.

The initial terms of the *caraṇas* (rows) may be supposed to be (i) 7, 12, 17, 22 or (ii) 4, 11, 18, 25 or (iii) 1, 10, 19, 28, the common-difference being unity in each case. The corresponding squares are shown in Figs. 43-45.

7	15	22	20
23	19	8	14
10	12	25	17
24	18	9	13

Total = 64

Fig. 43

4	14	25	21
26	20	5	13
7	11	28	18
27	19	6	12

Total = 64

Fig. 44

1	13	28	22
29	21	2	12
4	10	31	19
30	20	3	11

Total = 64

Fig. 45

Construction of Irregular Series

Method 1: Nārāyaṇa gives the following rule for the determination of irregular series to be used for filling a square with a give total:

"For the determination of the *caraṇas* ("rows") assume the first term and the common-difference optionally. First write down the initial term and then add to it successively the product of the common-difference and the number of cells in a row, and do so as many times as the number of rows less one. The series thus formed is the *mukhapaṅkti* ("the optionally assumed series of initial terms"). To the last term of this series add the first term together with the product of the common-difference into the number of rows minus one, and multiply by half the number of rows: this is the *mukhaphala* ("the total corresponding to the assumed series"). The desired total minus the *mukhaphala* is the *kṣepaphala* ("the total for the numbers to be interpolated"). Now determine the first term and the common-difference of a series in A.P. whose number of terms is equal to the number of rows and whose sum is equal to the *kṣepaphala*. Add the successive terms of the series thus obtained to the corresponding terms of the *mukhapaṅkti* ("the optionally assumed series of initial terms"). Thus will be determined the *caraṇas* for all magic squares"[25].

Example from Nārāyaṇa:

(i) Determine the *caraṇas* for a 4×4 magic square with total 40.

Optionally assume a series whose first term is 1 and the common-difference is 1. When the terms are placed in rows of four, the initial terms of the successive rows (i.e., *mukhapaṅkti*) are:

1, 5, 9, 13.

Since the number of *caraṇas* is 4,

$$\textit{mukhaphala} = \frac{4}{2}\,[13+1 + (4-1).1] = 34$$

$$\textit{kṣepaphala} = 40 - 34 = 6$$

Now, if A be the first term, and D, the common-difference and 6 the sum of an A.P. of 4 terms, we must have

$$\frac{6 - \frac{4}{2}(4-1)\,D}{4} = A$$

$\therefore\ A=0,\ -3, \ldots\ldots$

and $D=1,\ 3,\ \ldots\ldots.$

For the solution ($A=0$, $D=1$), the series is: 0, 1, 2, 3.

For the solution ($A=-3$, $D=3$), the series is: -3, 0, 3, 6.

Therefore, the initial terms of the required *caraṇas* ("rows") are (1, 6, 11, 16) or (-2, 5, 12, 19).

(ii) Determine the *caraṇas* for the 4×4 square whose total is 64.

In this case, the *kṣepaphala* is $64 - 34 = 30$

so that $\frac{30 - 6D}{4} = A$

i.e. $A = 6, 3, 0, \ldots$

and $D = 1, 3, 5, \ldots$.

For the first solution,the series is (6, 7, 8, 9), for the second (3, 6, 9, 12) and for the third (0, 5, 10, 15). Therefore, the initial terms of the *caraṇas* are (7, 12, 17, 22) or (4, 11, 18, 25) or (1, 10, 19, 28).

The squares may now be constructed by the method of the knight's move.

Method 2[26]: "Divide the *kṣepaphala* ("total of numbers to be interpolated") by the *caraṇa* ("number of cells in a row"). The quotient increased by unity becomes the "*gaccha*"[27], provided the remainder is zero or equal to half the *caraṇa*. If the remainder is otherwise, the magic square is not possible. Add to the first and the second halves of the *mukhapaṅkti* respectively zero and half the *kṣepaphala* or these increased and decreased by unity successively. Thus will be determined the initial terms of the *caraṇas* in the cases of *samagarbha* and *viṣamagarbha* squares.

Examples from Nārāyaṇa

(i) To construct a 4×4 square with total 40.

Assuming the series of natural numbers, the *kṣepaphala* is $40 - 34 = 6$. This divided by the *caraṇa*, i.e. $6 \div 4$, gives the quotient 1 and remainder 2. The construction of the square is thus possible, and $1 + 1 = 2$ squares may be obtained. The *mukhapaṅkti* is 1, 5, 9, 13. Half of the *kṣepaphala* $= 3$

The numbers to be interpolated are, therefore, 0 and 3, or adding and subtracting unity, 1 and 2. Thus, adding these to the respective halves of the *mukhapaṅkti*, we get:

0	3	1	5	12	16
1	2	2	6	11	15

Interpolators | Initial terms of the rows[28]

Thus, the numbers to be filled in the square are:

1, 2, 3, 4	or	2, 3, 4, 5
5, 6, 7, 8		6, 7, 8, 9
12, 13, 14, 15		11, 12, 13, 14
16, 17, 18, 19		15, 16, 17, 18

and the corresponding squares are as shown in Figs. 46 and 47.

1	8	16	15
17	14	2	7
4	5	19	12
18	13	3	6

Total = 40

Fig. 46

2	9	15	14
16	13	3	8
5	6	18	11
17	12	4	7

Total = 40

Fig. 47

(ii) To construct a 4×4 square with total 64.

Here, as before the *kṣepaphala* $= 64 - 34 = 30$.

This divided by the *caraṇa*, i.e., $30 \div 4$ gives the quotient 7 and remainder 2. Thus, the square is possible and $7 + 1 = 8$ different squares may be obtained.

As before, the *mukhapaṅkti* is 1, 5, 9, 13. Half the *kṣepaphala* is 15. The numbers to be interpolated are 0, 15, or adding and subtracting unity successively to get 8 different pairs we have:

0	15
1	14
2	13
3	12
4	11
5	10
6	9
7	8

Adding these pairs to the respective halves of the *mukhapaṅkti* (1, 5, 9, 13), we get the following 8 sets for the initial terms of the rows:

1	5	24	28
2	6	23	27
3	7	22	26
4	8	21	25
5	9	20	24
6	10	19	23
7	11	18	22
8	12	17	21

Eight squares may now be constructed as before.

Change of Squares

"Construct a magic square of the type desired. Subtract its total from the given total, and divide by the number of cells in a line. On adding the quotient to the numbers in the cells of that square will be obtained the required square"[29].

Thus, to transform the 4×4 magic square of Fig. 22, with total 34, into another with total 100, one has simply to add $(100-34)/4$, i.e., 33/2 to the numbers in the cells of that magic square.

Construction by Superposition

"Construct two *samagarbha* squares, one called *chādaka* ("covering one") and the other called *chādya* ("one to be covered"). The superposition is to be made in the manner of folding the palms of the hands. Form a series with an optional first term and an optional common-difference and with as many terms as the "number"[30] of the square; this is the *mūlapaṅkti* ("basic series"). With another first term and common-difference form another series: this is called *parapaṅkti*. Multiplying the terms of the *parapaṅkti* by the quotient obtained on dividing the given total minus the sum of the *mūlapaṅkti* by the sum of the *parapaṅkti*, is produced the progression which is called *guṇapaṅkti* ("product-series"). Divide the *mūlapaṅkti* and the *guṇapaṅkti* by turning each upon itself, so that each part will have terms equivalent to half the "number" of the square. The numbers are written down vertically, one above the other, and directly in the *chādaka* ("covering one") and in another fashion (i.e., horizontally and inversely) in the *chādya* ("one to be covered"). In the first, fill thus successively half the rows and in the second half the columns[31]: Fill the other half of each square in the contrary way. This method of constructing magic squares by superposition is taught by the son of Nṛhari (i.e., by Nārāyaṇa)"[32].

Examples from Nārāyaṇa

(i) To construct a 4×4 magic square with total 40.

Assume the *mūlapaṅkti* to be 1, 2, 3, 4

Let the *parapaṅkti*[33] be 0, 1, 2, 3

The multiplier $= (40 - 10)/6 = 5$

∴ The *gunapaṅkti* is 0, 5, 10, 15.

Writing the *mūlapaṅkti* and *guṇapaṅkti* by turning them upon themselves, we get

1	2		0	5	
		and			respectively.
4	3		15	10	

Taking the first set, placing it vertically and then filling with it the horizontal half of a 4×4 square, we get fig. 48.

2	3	2	3
1	4	1	4

Fig. 48

Then filling the other half with the same numbers in the inverse order, we get the *chādaka* (Fig. 49).

(A)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

Chādyaka
Fig. 49

In the same way, filling horizontally with the second set, we get the *chādya* (Fig. 50).

(B)

5	0	10	15
10	15	5	0
5	0	10	15
10	15	5	0

Chādya
Fig. 50

Then, folding (A) over (B) and adding the numbers, we get the required square with total 40 (Fig. 51).

(AB)

8	2	13	17
14	16	9	1
7	3	12	18
11	19	6	4

Total = 40

Aliter. Or, if we take the *mūlapaṅkti* as before but the *parapaṅkti* as 1, 2, 3, 4, the multiplier is (40 − 10)/10 = 3, so that the *guṇapaṅkti* is 3, 6, 9, 12. Thus we get

1	2		3	6
		and		
4	3		12	9

and the corresponding squares are

(A′)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

Fig. 52

(B′)

6	3	9	12
9	12	6	3
6	3	9	12
9	12	6	3

Fig. 53

(A′B′)

9	5	12	14
13	13	10	4
8	6	11	15
10	16	7	7

Total = 40

Fig 54 (vide ref. 34)

(ii) To construct a 4 × 4 square with total 64.

Here, taking

the *mūlapaṅkti* as 1, 2, 3, 4

and the *parapaṅkti* as 0, 1, 2, 3

the multiplying factor is $\dfrac{64-(1+2+3+4)}{(0+1+2+3)} = 9$

∴ the *guṇapankti* is 0, 9, 18, 27.

Arranging, we have correspondingly

1	2	0	9
4	3	27	18

(A)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

Fig. 55

(B)

9	0	18	27
18	27	9	0
9	0	18	27
18	27	9	0

Fig. 56

(AB)

12	2	21	29
22	28	13	1
11	3	20	30
19	31	10	4

Total = 64

Fig. 57

as before (Figs. 55-57).

The above method of constructing squares was rediscovered in Europe by M. de la Hire (1705), and is now attributed to him.

(iii) to construct a 8×8 square with total 260.

Let the *mūlapaṅkti* be 1, 2, 3, 4, 5, 6, 7, 8,

and the *parapaṅkti* 0, 1, 2, 3, 4, 5, 6, 7.

$$\text{The multiplying factor} = \frac{260 - \frac{1}{2} \cdot 8\,(8+1)}{\frac{1}{2} \cdot 7\,(7+1)} = 8$$

The *guṇapaṅkti* is 0, 8, 16, 24, 32, 40, 48, 56.

Breaking up the *mūlapaṅkti* and *guṇpaṅ* into halves and writing them by turning upon themselves we have

(a)

1	2	3	4
8	7	6	5

and

(b)

0	8	16	24
56	48	40	32

Hence, the preliminary squares are as shown in Figs. 58 and 59.

4	5	4	5	4	5	4	5
3	6	3	6	3	6	3	6
2	7	2	7	2	7	2	7
1	8	1	8	1	8	1	8
5	4	5	4	5	4	5	4
6	3	6	3	6	3	6	3
7	2	7	2	7	2	7	2
8	1	8	1	8	1	8	1

Fig. 58

24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0

Fig. 59

Superposing these two as in the hinge, we get Fig. 60.

60	53	44	37	4	13	20	29
3	14	19	30	59	54	43	38
58	55	42	39	2	15	18	31
1	16	17	32	57	56	41	40
61	52	45	36	5	12	21	28
6	11	22	27	62	51	46	35
63	50	47	34	7	10	23	26
8	9	24	25	64	49	48	33

Total = 260

Fig. 60 (vide ref. 35)

Second Method: "In as many 4 × 4 squares as are present in the *samagarbha* (4n × 4n square) such as 8 × 8 square, etc., write the numbers produced in the series, as in the method of the 4 × 4 square by right and left (knight's) moves. Thus is said the easy method of constructing *samagarbha* (4n × 4n) squares such as 8 × 8 square, etc.[36]

Example from Nārāyaṇa

To construct a 8 × 8 square with total 260.

It is easily seen that the series of natural numbers from 1 to 64 is to be used. Writing the numbers 1 to 64 in groups of 4, we have

	1	8	9	16	48	41	40	33	
	2	7	10	15	47	42	39	34	
(I)	3	6	11	14	46	43	38	35	(III)
	4	5	12	13	45	44	37	36	
	32	25	24	17	49	56	57	64	
(II)	31	26	23	18	50	55	58	63	(IV)
	30	27	22	19	51	54	59	62	
	29	28	21	20	52	53	60	61	

Interchanging the figures in the third and fourth columns, as in the method of filling 4×4 squares, we get

	1	8	16	9	48	41	33	40	
I	2	7	15	10	47	42	34	39	III
	3	6	14	11	46	43	35	30	
	4	5	13	12	45	44	36	37	
	32	25	17	24	49	56	64	57	
	31	26	18	23	50	55	63	58	
II	30	27	19	22	51	54	62	59	IV
	29	28	20	21	52	53	61	60	

Taking the first rows of I and II to fill the first 4×4 square, the second rows to fill the second and so on, we get Fig. 61.

1	32			2	31		
		8	25			7	26
16	17			15	18		
		9	24			10	23
4	29			3	30		
		5	28			6	27
13	20			14	19		
		12	21			11	22

Fig. 61

Then taking the first rows of III and IV to fill the remaining cells of the first 4×4 square, the second rows to fill the remaining cells of the second 4×4 square and so on, we get Fig. 62.

1	32	49	48	2	31	50	47
56	41	8	25	55	42	7	26
16	17	64	33	15	18	63	34
57	40	9	24	58	39	10	23
4	29	52	45	3	30	51	46
53	44	5	28	54	43	6	27
13	20	61	36	14	19	62	35
60	37	12	21	59	38	11	22

Total = 260
Fig 62 (vide ref. 37)

Viṣamagarbha Squares

Nārāyaṇa gives two methods of construction of the (4n+2) × (4n+2) squares.

First method: This method is described by Nārāyaṇa thus:

"The measure of the *śliṣṭa*[38] cells is half of half the "number of the square" minus one. All over the square write down the numbers in connected cells in the direct and inverse order, one below the other (in rows). The numbers standing in the middle two columns above and below the middle two rows, excepting those in the last but one column below, should be interchanged (by one place anticlockwise turning). Then the two middle numbers in the extreme right of the right half of the square should be interchanged with the corresponding ones of the left half of the square, which lie attached to the diagonal. Finally the numbers of the *śliṣṭa* cells in the upper and lower halves of the square should be interchanged symmetrically. Such is the procedure of filling the cells with numbers by the method of *śliṣṭa* cells. The numbers in the cells attached to the diagonal in the right half of the square should be left as they are. Others may be interchanged if necessary to make up the total. This is the method of constructing the *Viṣamagarbha-bhadra* taught by Nārāyaṇa"[39].

Examples from Nārāyaṇa

(i) To construct a 6×6 square with total 111.

It is easily seen that the series of natural numbers from 1 to 36 is to be used.

The measure of the *śliṣṭa* cells = $(3-1)/2=1$.

The numbers 1 to 36 are placed in the 6×6 square in the direct and inverse order, as in Fig. 63.

1	2*	3	4	5*	6
12*	11	10	9	8	7*
13*	14	15	16	17	18*
24*	23	22	21	20	19*
25*	26	27	28	29	30*
36	35*	34	33	32*	31

Fig. 63

There is only one *śliṣṭa* cell in each half row. These lie at the ends and are marked by asterisks.

The numbers in the two middle columns lying above and below the two middle rows excepting those in the last but one co-column below are interchanged (by one place anticlockwise turning) as shown below. The numbers in the extreme right cells of the two middle rows are interchanged with the corresponding ones of the left half of the square. This gives Figs. 64 and 65.

1	*	4	33	*	6
*	11	9	28	8	*
*	14	15	16	17	18*
*	23	22	21	20	19*
*	26	27	10	29	*
36	*	34	3	*	31

Fig. 64

1	*	4	33	*	6
*	11	9	28	8	*
*	14	18	16	17	15*
*	23	19	21	20	22*
*	26	27	10	29	*
36	*	34	3	*	31

Fig. 65

The numbers standing in the *ślista* cells above are then interchanged with the corresponding ones below, giving Fig. 66.

1	35	4	33	32	6
25	11	9	28	8	30
24	14	18	16	17	22
13	23	19	21	20	15
12	26	27	10	29	7
36	2	34	3	5	31

Total = 111

Fig. 66

(ii) To fill a 10×10 square with the natural numbers 1 to 100.

In this case, the total $= \dfrac{100 \times 101}{2 \times 10} = 505.$

Placing the numbers 1 to 100 in a 10×10 square, we get Fig. 67.

1	2	3	4	5	6	7	8	9	10
20	19	18	17	16	15	14	13	12	11
21	22	23	24	25	26	27	28	29	30
40	39	38	37	36	35	34	33	32	31
41	42	43	44	45	46	47	48	49	50
60	59	58	57	56	55	54	53	52	51
61	62	63	64	65	66	67	68	69	70
80	79	78	77	76	75	74	73	72	71
81	82	83	84	85	86	87	88	89	90
100	99	98	97	96	95	94	93	92	91

Fig. 67

Interchanging the numbers in the middle columns as directed and also those in the extreme right cells of the two middle rows with the corresponding ones of the left half of the square, we get Fig. 68.

1	*	*	4	6	95	7	*	*	10
*	19	*	17	15	86	14	*	12	*
*	*	23	24	26	75	27	28	*	*
*	*	38	37	35	66	34	33	*	*
*	*	43	44	50	46	47	48	*	* 45
*	*	58	57	51	55	54	53	*	* 56
*	*	63	64	65	36	67	68	*	*
*	*	78	77	76	25	74	73	*	*
*	82	*	84	85	16	87	*	89	*
100	*	*	97	96	5	94	*	*	91

Fig. 68

Then interchanging the numbers in the *śliṣṭa* cells (marked by asterisks) as before we have the required square (Fig. 69).

1	99	98	4	6	95	7	93	92	10
81	19	83	17	15	86	14	88	12	90
80	79	23	24	26	75	27	28	72	71
61	62	38	37	35	66	34	33	69	70
60	59	43	44	50	46	47	48	52	56
41	42	58	57	51	55	54	53	49	45
40	39	63	64	65	36	67	68	32	31
21	22	78	77	76	25	74	73	29	30
20	82	18	84	85	16	87	13	89	11
100	2	3	97	96	5	94	8	9	91

Total = 505

Fig. 69

(iii) To construct a 14×14 square with the series of natural numbers.

The numbers filled continuously in a 14×14 square and then interchanged according to Nārāyaṇa's rule give Figs. 70 and 71.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
28	27	26	25	24	23	22	21	20	19	18	17	16	15
29	30	31	32	33	34	35	36	37	38	39	40	41	42
56	55	54	53	52	51	50	49	48	47	46	45	44	43
57	58	59	60	61	62	63	64	65	66	67	68	69	70
84	83	82	81	80	79	78	77	76	75	74	73	72	71
85	86	87	88	89	90	91	92	93	94	95	96	97	98
112	111	110	109	108	107	106	105	104	103	102	101	100	99
113	114	115	116	117	118	119	120	121	122	123	124	125	126
140	139	138	137	136	135	134	133	132	131	130	129	128	127
141	142	143	144	145	146	147	148	149	150	151	152	153	154
168	167	166	165	164	163	162	161	160	159	158	157	156	155
169	170	171	172	173	174	175	176	177	178	179	180	181	182
196	195	194	193	192	191	190	189	188	187	186	185	184	183

Key-square

Fig. 70

1	195	194	193	5	6	8	189	9	10	186	185	184	14
169	27	171	172	24	23	21	176	20	19	179	180	16	182
168	167	31	165	33	34	36	161	37	38	158	40	156	155
141	142	143	53	52	51	49	148	48	47	46	152	153	154
140	139	138	60	61	62	64	133	65	66	67	129	128	127
113	114	115	81	80	79	77	120	76	75	74	124	125	126
112	111	110	88	89	90	98	92	93	94	95	101	100	106
85	86	87	109	108	107	99	105	104	103	102	96	97	91
84	83	82	116	117	118	119	78	121	122	123	73	72	71
57	58	59	137	136	135	134	63	132	131	130	68	69	70
56	55	54	144	145	146	147	50	149	150	151	45	44	43
29	30	166	32	164	163	162	35	160	159	39	157	41	42
28	170	26	25	173	174	175	22	177	178	18	17	181	15
196	2	3	4	192	191	190	7	188	187	11	12	13	183

Total = 1379

Fig. 71

Remarks

It will be observed that when the series employed is in A.P., the squares are constructed by making the minimum interchanges expressly stated by Nārāyaṇa. That this is so in all cases is illustrated by the 18 × 18 squares constructed according to this method (Figs. 72 and 73).

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
36	35	34	33	32	31	30	29	28	27	26	25	24	23	22	21	20	19
37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
72	71	70	69	68	67	66	65	64	63	62	61	60	59	58	57	56	55
73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
108	107	106	105	104	103	102	101	100	99	98	97	96	95	94	93	92	91
109	110	111	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126
144	143	142	141	140	139	138	137	136	135	134	133	132	131	130	129	128	127
145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162
180	179	178	177	176	175	174	173	172	171	170	169	168	167	166	165	164	163
181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198
216	215	214	213	212	211	210	209	208	207	206	205	204	203	202	201	200	199
217	218	219	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234
252	251	250	249	248	247	246	245	244	243	242	241	240	239	238	237	236	235
253	254	255	256	257	258	259	260	261	262	263	264	265	266	267	268	269	270
288	287	286	285	284	283	282	281	280	279	278	277	276	275	274	273	272	271
289	290	291	292	293	294	295	296	297	298	299	300	301	302	303	304	305	306
324	323	322	321	320	319	318	317	316	315	314	313	312	311	310	309	308	307

Key-square

Fig. 72

When, however, the series to be used is a broken series, other changes have to be made. For instance, to construct a 6×6 square with total 132, one may take the series of initial terms 2, 9, 16, 23, 30, 37 and common-difference 1. Proceeding according to the rule, we get Fig. 74.

But in this square, the third and fourth rows do not have the desired total. We, therefore, replace the initial numbers 28 and 16 of the third and fourth rows by 29 and 15 respectively and thus we get the magic square shown in Fig. 75.

1	323	322	321	320	6	7	8	10	315	11	12	13	311	310	309	308	18
289	35	291	292	293	31	30	29	27	298	26	25	24	302	303	304	20	306
288	287	39	285	284	42	43	44	46	279	47	48	49	275	274	52	272	271
253	254	255	69	257	67	66	65	63	262	62	61	60	266	58	268	269	270
252	251	250	249	77	78	79	80	82	243	83	84	85	86	238	237	236	235
217	218	219	220	104	103	102	101	99	226	98	97	96	95	231	232	233	234
216	215	214	213	113	114	115	116	118	207	119	120	121	122	202	201	200	199
181	182	183	184	140	139	138	137	135	190	134	133	132	131	195	196	197	198
180	179	178	177	149	150	151	152	162	154	155	156	157	158	166	165	164	172
145	146	147	148	176	175	174	173	163	171	170	169	168	167	159	160	161	153
144	143	142	141	185	186	187	188	189	136	191	192	193	194	130	129	128	127
109	110	111	112	212	211	210	209	208	117	206	205	204	203	123	124	125	126
108	107	106	105	221	222	223	224	225	100	227	228	229	230	94	93	92	91
73	74	75	76	248	247	246	245	244	81	242	241	240	239	87	88	89	90
72	71	70	256	68	258	259	260	261	64	263	264	265	59	267	57	56	55
37	38	286	40	41	283	282	281	280	45	278	277	276	50	51	273	53	54
36	290	34	33	32	294	295	296	297	28	299	300	301	23	22	21	305	19
324	2	3	4	5	319	318	317	316	9	314	313	312	14	15	16	17	307

Total = 2925

Fig. 73

In this magic square, no number has been repeated. Replacement of the numbers 17 and 27 (standing in the third and fourth rows) by 18 and 26, or 20 and 24 by 21 and 23, or 26 and 18 by 27 and 17 will also yield magic squares with total 132, but there will be repetitions of two numbers.

Nārāyaṇa, however, gives Fig. 74 as a 6×6 magic square with total 132. But as pointed out above, it is truly speaking not a magic square.

2	41	5	39	38	7
30	13	11	33	10	35
28	17	21	19	20	26
16	27	23	25	24	18
14	31	32	12	34	9
42	3	40	4	6	37

Fig. 74

2	41	5	39	38	7
30	13	11	33	10	35
29	17	21	19	20	26
15	27	23	25	24	18
14	31	32	12	34	9
42	3	40	4	6	37

Total = 132

Fig. 75

If we use the series of initial terms 1, 7, 13, 26, 32, 38 and common-difference 1, and proceed as above, we shall get Fig. 76.

1	42	4	40	39	6
32	11	9	35	8	37
31	14	18	16	17	29
13	30	26	28	27	15
12	33	34	10	36	7
43	2	41	3	5	38

Fig. 76

Here also, the third and fourth rows do not have the desired total. But if we replace the numbers 31 and 13, in those rows, by 38 and 6, or 14 and 30 by 21 and 23, or 17 and 27 by 24 and 20, or 29 and 15 by 36 and 8 respectively, we shall get 4 magic squares with total 132. In two of these magic squares there will be no repetition of numbers, but in the other two there will be repetition of numbers.

Second method: "In the *Viṣamagarbha* squares such as 6×6, etc. the two middle lines of cells (both horizontal and vertical) are called *Pīṭha*. Fill the cells of the square with the numbers (of the given or chosen series) in the direct order. Reverse the number in the cells of each diagonal. Then interchange the numbers lying at the north-east corner between the diagonal and the *Pīṭha* with the numbers in the (corresponding) opposite cells. Then interchange the two numbers at the south *Pīṭha*, and also those at the west *Pīṭha*. Thus will be obtained the desired total in the horizontal and vertical outskirts of the square. The interchange of the numbers in the other cells should be made as required in order to make up the total by noting the deficit or excess from it"[40].

Examples (i): To construct a 6×6 square with the series of natural numbers.

The numbers are placed in the square in the direct order as in Fig. 77.

In the above, the *Pīṭhas* ("central rows and columns") are marked by thick lines. The directions are indicated by the letters E, N, W and S. The numbers in the diagonal cells are reversed. The numbers 2 and 7 lying at the north-east corner between the diagonal cells and the *Pīṭha* cells are interchanged with the numbers 32 and 12, respectively, which lie in the corresponding opposite cells. Then the numbers 18 and 24 at the south *Pīṭha* are interchanged; so also are interchanged the numbers 33 and 34 at the west *Pīṭha*. Thus, we have Fig. 78,

E

N (left), S (right)

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

W

Fig. 77

36	32	3	4	5	31
12	29			26	7
13		22	21		24
19		16	15		18
25	11			8	30
6	2	34	33	35	1

Fig. 78

in which the totals of the bounding rows and columns are as desired. The sums of the diagonal cells are also as desired. The other numbers should now be interchanged by trial to get the desired total 111. The squares shown in Figs. 79 and 80 result.

36	32	3	4	5	31
12	29	27	10	26	7
13	17	22	21	14	24
19	20	16	15	23	18
25	11	9	28	8	30
6	2	34	33	35	1

Total = 111

Fig. 79

36	32	3	4	5	31
12	29	9	28	26	7
13	14	22	21	17	24
19	23	16	15	20	18
25	11	27	10	8	30
6	2	34	33	35	1

Total = 111

Fig. 80

(ii) To construct a 10 × 10 square with the series of natural numbers.

The above process gives the square shown in Fig. 81.

100	92	93	94	5	6	7	8	9	91
20	89	88	14	16	15	87	83	82	11
30	29	78	77	75	26	74	73	22	21
40	39	63	67	65	66	64	38	32	31
41	49	48	54	56	55	57	43	42	60
51	52	53	47	46	45	44	58	59	50
61	62	33	37	35	36	34	68	69	70
71	72	28	27	25	76	24	23	79	80
81	19	18	84	86	85	17	13	12	90
10	2	3	4	96	95	97	98	99	1

Fig. 81

VIṢAMA SQUARES

Nārāyaṇa gives two methods for the construction of the "*Viṣamabhadra* ("Odd squares"). The first of these is Nārāyaṇa's own method, the method of superposition, which was rediscovered in the west by M de la Hire (1705). The second method seems to have been known in India before Nārāyaṇa.

First method: "Determine the *mūlapaṅkti* and *guṇapaṅkti* in the way indicated before. The first term of the former should be placed in the centre cell of the top row of the first of the (*chādya* and *chādaka*) squares. Beneath it should be written down vertically the successive terms of the series. The other columns should be filled similarly, so that the numbers in the top row are in order. In the same way, beginning with the first term of the second series fill up the second square. The method of superposition of the *chādya* ("one to be covered") and *chādaka* ("covering one") is as before"[41].

Examples from Nārāyaṇa

(i) To construct a 3×3 square with total 24.

Assume the *mūlapaṅkti* ("basic series") to be 1, 2, 3.

Let the *parapaṅkti* ("second series") be 0, 1, 2.

Then the multiplying factor is $\frac{24-(1+2+3)}{(0+1+2)} = 6$

Therefore, the *guṇapaṅkti* is 0, 6, 12.

Now, filling the *chādaka* ("one to be covered") square with the *mūlapaṅkti* and *chādaka* ("covering one") with the *guṇapaṅkti* as directed in the rule, we get Figs. 82 and 83.

Superposing these as in a hinge, we have the required square (Fig. 84).

3	1	2
1	2	3
2	3	1

Fig. 82

12	0	6
0	6	12
6	12	0

Fig. 83

9	1	14
13	8	3
2	15	7

Total = 24

Fig. 84

(ii) To construct a 5×5 square with total 90.

Let the *mūlapankti* be 1, 2, 3, 4, 5

Also let the *parapankti* be 1, 2, 3, 4, 5

Then the multiplier is $\frac{90-(1+2+3+4+5)}{1+2+3+4+5} = 5$

Therefore, the *guṇapankti* is 5, 10, 15, 20, 25.

Filling the squares as before, we have Figs. 85-87.

4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

Chādya
Fig. 85

20	25	5	10	15
25	5	10	15	20
5	10	15	20	25
10	15	20	25	5
15	20	25	5	10

Chādaka
Fig. 86

19	15	6	27	23
25	16	12	8	29
26	22	18	14	10
7	28	24	20	11
13	9	30	21	17

Total = 90
Fig. 87

5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3

Chādya

(iii) To construct a 7×7 square with total 238.

Here, taking the *mūlapaṅkti* as 1, 2, 3, 4, 5, 6, 7, and the *parapaṅkti* as 0, 1, 2, 3, 4, 5, 6, the *guṇapaṅkti* is 0, 10, 20, 30, 40, 50, 60. The squares obtained as above are shown in Figs. 88-90.

40	50	60	0	10	20	30
50	60	0	10	20	30	40
60	0	10	20	30	40	50
0	10	20	30	40	50	60
10	20	30	40	50	60	0
20	30	40	50	60	0	10
30	40	50	60	0	10	20

Chādaka

Fig. 89

35	26	17	1	62	53	44
46	37	21	12	3	64	55
57	41	32	23	14	5	66
61	52	43	34	25	16	7
2	63	54	45	36	27	11
13	4	65	56	47	31	22
24	15	6	67	51	42	33

Total = 238

Fig. 90

Second method: "In the first cell of a middle line (of cells) write the first term of the series of numbers, and in the cell beside the opposite cell of the same line (write) the next number. Then, in the cells lying along the shorter diagonal from that write the following numbers. (On reaching an extremity) continue the filling beginning with the cell of the opposite line which will be diagonally in front (considering the square to be rolled on a cylinder). When the next diagonal cell is found to be already filled up, begin from the cell behind and fill successively (in the same way). In the *viṣamabhadra* there will be eight varieties"[42].

Examples from Nārāyaṇa

(i) To construct a 3×3 square with the series of natural numbers.

Writing 1 at the top of the middle line (column), the 2 in the last cell of the next column and proceeding diagonally upwards we have Fig. 91.

8	1	6
3	5	7
4	9	2

Total = 15

Fig. 91

In the above, whenever a block occurs, we begin with the cell underneath.

Another filling would be as per Fig. 92.

As the filling can be started by placing the first term in any one of the four centre cells of the outskirts, there will be altogether 8 different squares, as stated by Nārāyaṇa.

(ii) To construct a 3×3 square with total 24.

Nārāyaṇa uses an irregular series for filling up the square. According to the method for finding out such series, we get 3, 7, 11 as the initial terms of the *caraṇas*, the common difference being 1. The numbers to be filled are, therefore,

3, 4, 5
7, 8, 9
11, 12, 13

Hence the magic square is as in Fig. 93.

6	1	8
7	5	3
2	9	4

Total = 15

Fig. 92

7	5	12
13	8	3
4	11	9

Total = 24

Fig. 93

Note: In this and the following squares, the filling begins from the extreme right cell of the middle row.

(iii) To construct a 5×5 square with total 90.

Here, the initial terms of the *caraṇas* are found to be 4, 10, 16, 22, 28, the common-difference being unity. The square is shown in Fig. 94.

(iv) To construct a 7×7 square with total 238.

In this case, the initial terms of the *caraṇas* may be taken as 7, 15, 23, 31, 39, 47, 55, the common-difference being unity. The square is shown in Fig. 95.

16	14	7	30	23
24	17	10	8	31
32	25	18	11	4
5	28	26	19	12
13	6	29	22	20

Total = 90

Fig. 94

31	29	20	11	58	49	40
41	32	23	21	12	59	50
51	42	33	24	15	13	60
61	52	43	34	25	16	7
8	55	53	44	35	26	17
18	9	56	47	45	36	27
28	19	10	57	48	39	37

Total = 238

Fig. 95

The magic squares constructed by the above method are such that the sum of any two numbers that are geometrically equidistant from the centre is equal to twice the centre number. Such squares are called perfect by W.S. Andrews[43].

OTHER MAGIC FIGURES

Nārāyaṇa says: "With the help of 4×4 magic squares filled by natural numbers 1, etc. construct a magic rectangle or 4n×4n magic square. From it one can always construct other magic figures. Lines drawn through the corners in any desired way so as always to keep the number of cells the same give rise to the figures of *Vitāna* ("canopy"), *Maṇḍapa* ("altar"), *Vajra* ("diamond"), etc. Those are *Saṅkīrṇabhadra* ("other magic figures"). By the meeting together of lines between two cells and two diagonals are produced bases and uprights of triangle-pairs in all directions. Here, the triangles are filled with the numbers of a magic rectangle produced by 4n×4n squares, first in the direct order and then in the inverse order and so on. Such is the method of filling magic figures"[44].

Besides the three types of magic figures mentioned above, Nārāyaṇa has given rules for the construction of many other types of figures with illustration. These figures will be given and their peculiarities pointed out. The rules regarding their constructions will not be given, as they are apparent from the figures.

Vitāna ("canopy"): The figure is as shown in Fig. 96.

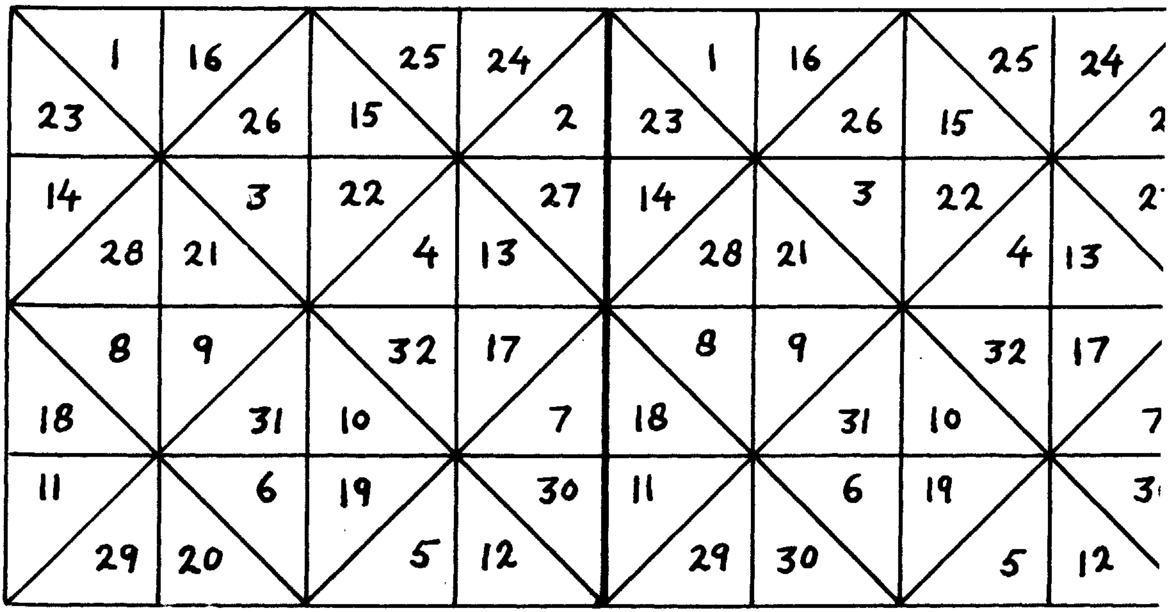
1	16	25	24	2	15	26	23
28	21	4	13	27	22	3	14
8	9	32	17	7	10	31	18
29	20	5	12	30	19	6	11

Vitāna or Canopy
Fig. 96

This is a rectangle constructed with the natural numbers 1 to 32 and consists of two 4×4 squares. The numbers are filled according to the method of 4n×4n squares given before[45]. It will be observed that the total of each row in the above is 132 and that of each column is 66.

For another magic rectangle constructed with the natural numbers 1 to 48 and consisting of three squares, see below (Fig. 105)

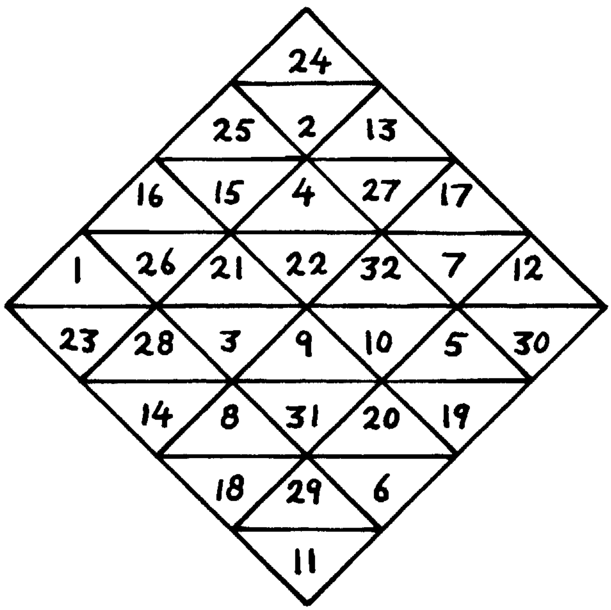
Maṇḍapa ("Altar"): The figure is



Maṇḍapa or Altar
Fig. 97

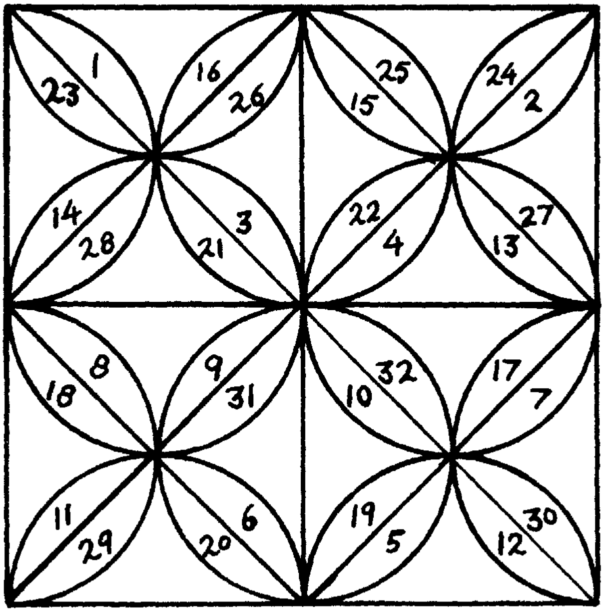
Here the numbers of the magic rectangle (Fig. 96) have been used by taking them successively in rows. Here, any set of eight numbers occurring together[46], horizontally, vertically or diagonally, gives the total 132. The eight numbers lying in a square have the same total 132. There is cylindrical symmetry, i.e., if the figure be rolled on a cylinder, any continuous eight numbers or those lying in a square give the total 132. It is easy to find 26 sets of eight numbers having the same total 132.

Vajra ("Diamond"): The figure (Fig. 98) is constructed from the magic rectangle in Fig. 96. Any eight numbers lying together in the same line, as well as the vertical diagonal, have the same sum 132. The sum of two horizontal rows, one in the upper half of the square and the other in the lower half, together containing eight numbers is 132. The sum of eight numbers lying in a small square is 132. In this case, it is easy to find 32 sets of eight numbers having the same total.



Vajra or Diamond
Fig. 98

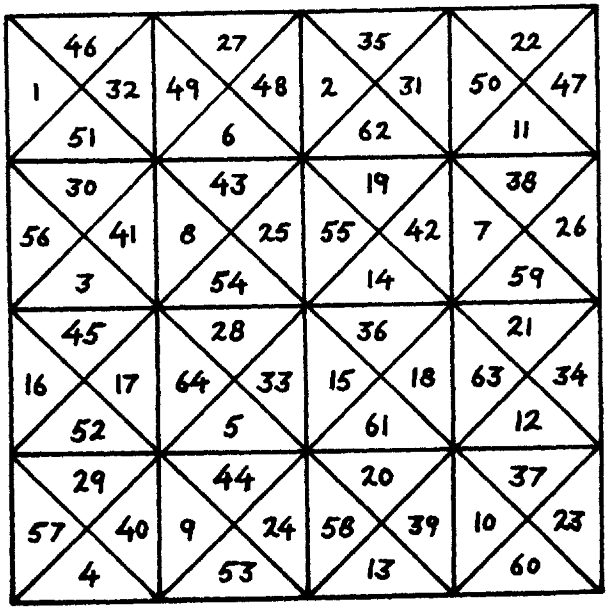
Padma ("Lotus"): The figure (Fig. 99) is constructed from the rectangle in Fig. 96. Any set of eight numbers taken vertically, horizontally (along lines side by side) or in any four leaves symmetrically situated give the same total 132. There is cylindrical symmetry. In this case, 32 sets of eight numbers having the same total can be easily picked out.



Padma or Lotus

Fig. 99

Vajra ("Diamond"): The *vajra* ("Diamond") (Fig. 100) uses the numbers of the 8 × 8 magic square in Fig. 62.

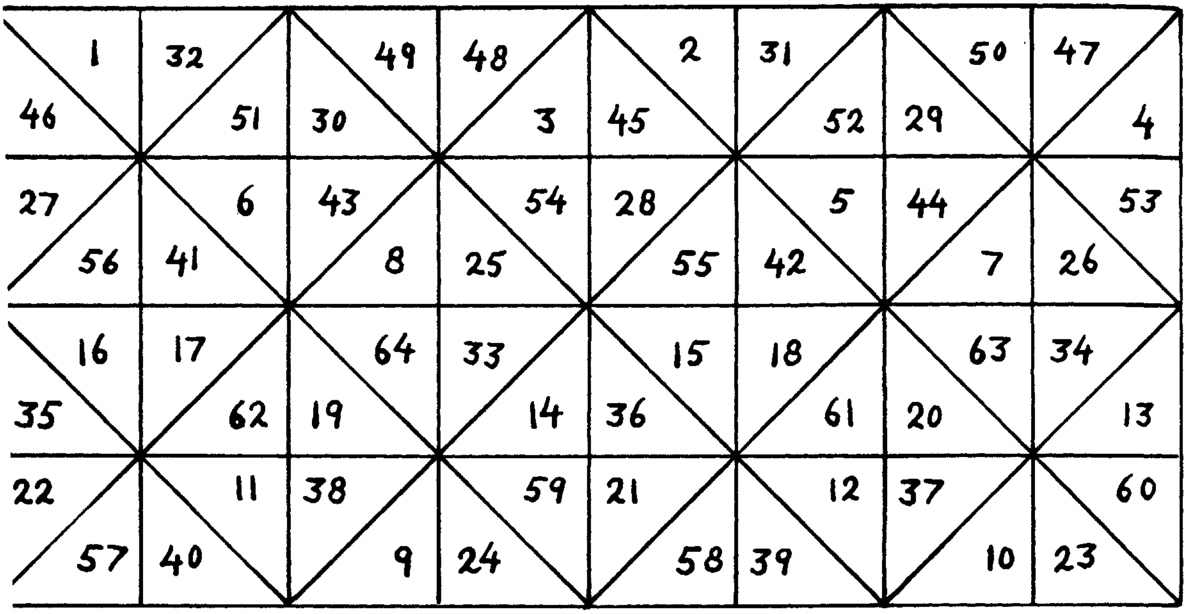


Vajra or Diamond

Fig. 100

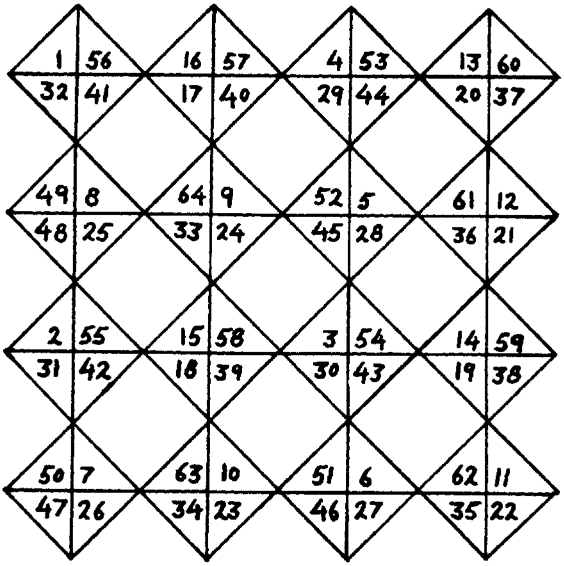
In the above groups of 16 numbers with the same total 520, groups of eight numbers with total 260 and also groups of four numbers with total 130 can be picked out easily. The sixteen numbers may be taken horizontally, vertically, and in two rings, etc. Groups of eight may be taken horizontally, vertically, diagonally, in rings, etc. Groups of four may be taken horizontally or vertically, as half rows or columns, in small squares, etc.

Maṇḍapa: The following *maṇḍapa* ("altar") is constructed by using the numbers of the 8 × 8 magic square in Fig. 62. It has groups of sixteen, eight and four numbers having equal totals, as in the *Vajra*.



Maṇḍapa
Fig. 101

Sarvatobhadra (*"Perfect magic figure"*): In this figure (Figs. 102 and 103) constructed from the 8 × 8 magic square in Fig.62, the totals of all four, eight and sixteen numbers are 130, 260 and 520 respectively. The figure is perfectly continuous.

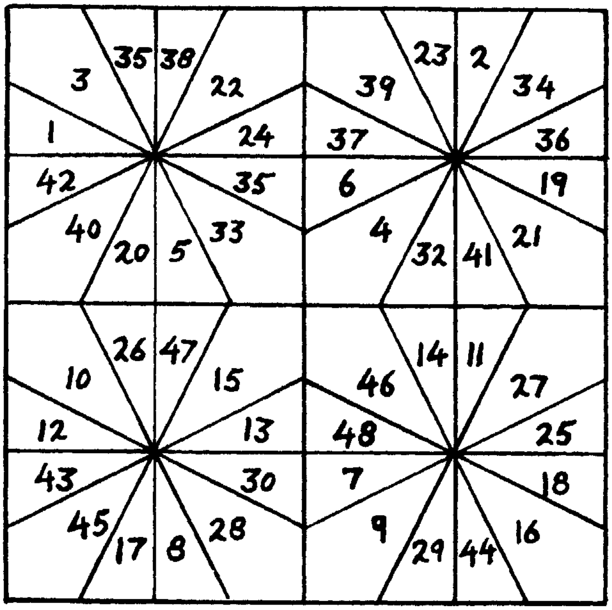


Sarvatobhadra
Fig. 102

1	56			16	57			4	53			13	60
32	41			17	40			29	44			20	37
49	8			64	9			52	5			61	12
48	25			33	24			45	28			36	21
2	55			15	58			3	54			14	59
31	42			18	39			30	43			19	38
50	7			63	10			51	6			62	11
47	26			34	23			47	27			35	22

Sarvatobhadra
Fig. 103

Dvādaśa-Kara ("Twelve-hands"): The figure (Fig. 104) is constructed by the numbers of the 12×4 rectangle (Fig. 105) using the numbers 1 to 48.



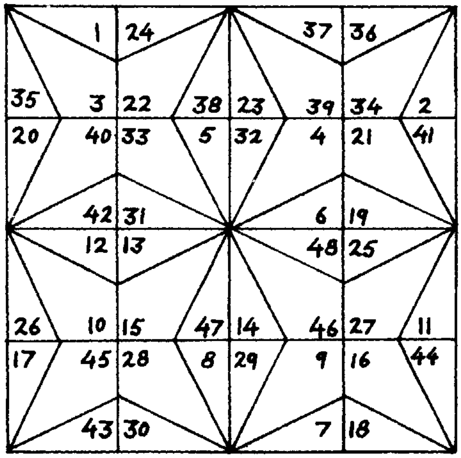
Dvādaśakara
Fig. 104

In the above figure all groups of 12, of 8 or of 4 numbers have equal totals, 294, 196 and 98 respectively.

1	24	37	36	2	23	38	35	3	22	39	34
42	31	6	19	41	32	5	20	40	33	4	21
12	13	48	25	11	14	47	26	10	15	46	27
43	30	7	18	44	29	8	17	45	28	9	16

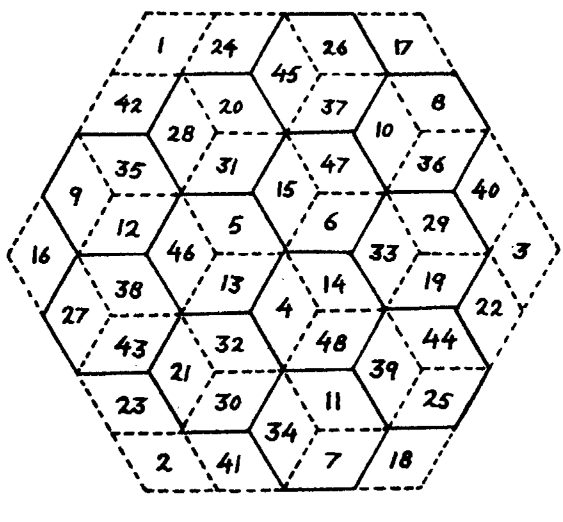
Fig. 105

Vajra Padma ("Diamond lotus"): The figure (Fig. 106) is constructed with the numbers of the 12×4 rectangle given above. In this figure, every group of four numbers whether occurring in a line or cells has the total 98, every group of eight numbers has the total 196 and every group of 12 numbers taken horizontally, vertically or in a circle has the total 294.



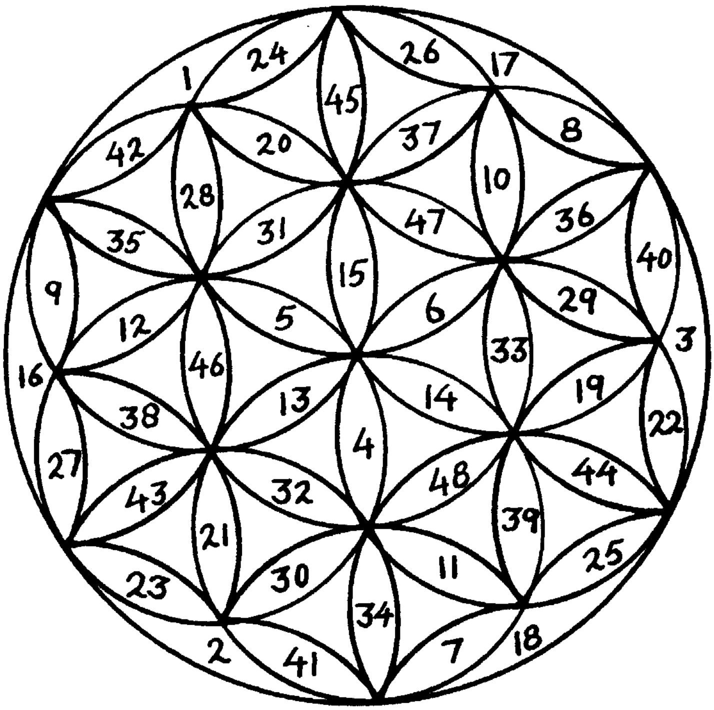
Vajra Padma
Fig. 106

Ṣaḍasra ("Hexagon"): The figure (Fig. 107) is constructed with the numbers of the 12×4 rectangle. In it every group of twelve numbers has the same sum 294.



Ṣaḍasra or Hexagon
Fig. 107

Padma Vṛtta ("Inscribed lotus"): The figure (Fig. 108) is constructed with the numbers of the 12 × 4 magic rectangle. Every group of twelve numbers has the same sum 294.



Padmavṛtta or Lotus circle

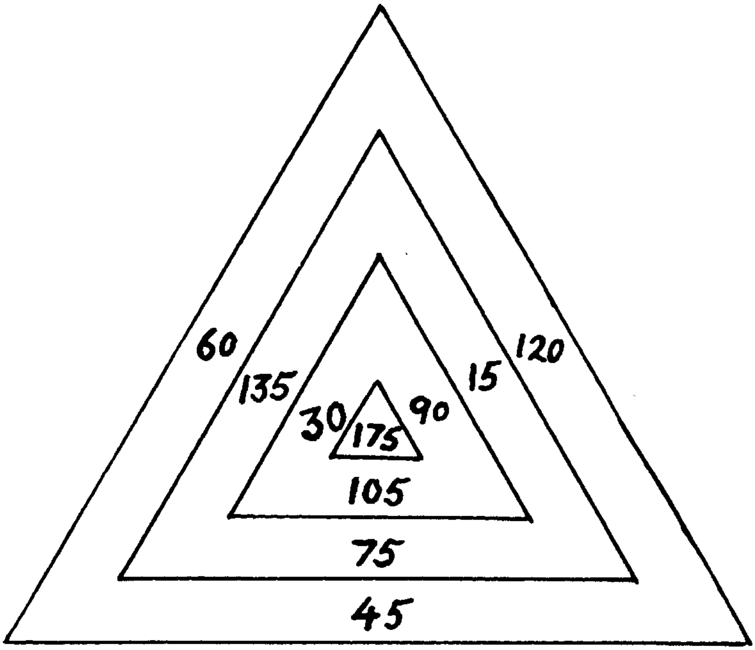
Fig. 108

Magic Triangle: Nārāyaṇa has proposed the problem of constructing a magic triangle with total 400. His magic triangle is constructed with the help of the numbers of a magic square whose total is 225.

120	15	90
45	75	105
60	135	30

Total = 225

Fig. 109



Total = 400
Fig. 110

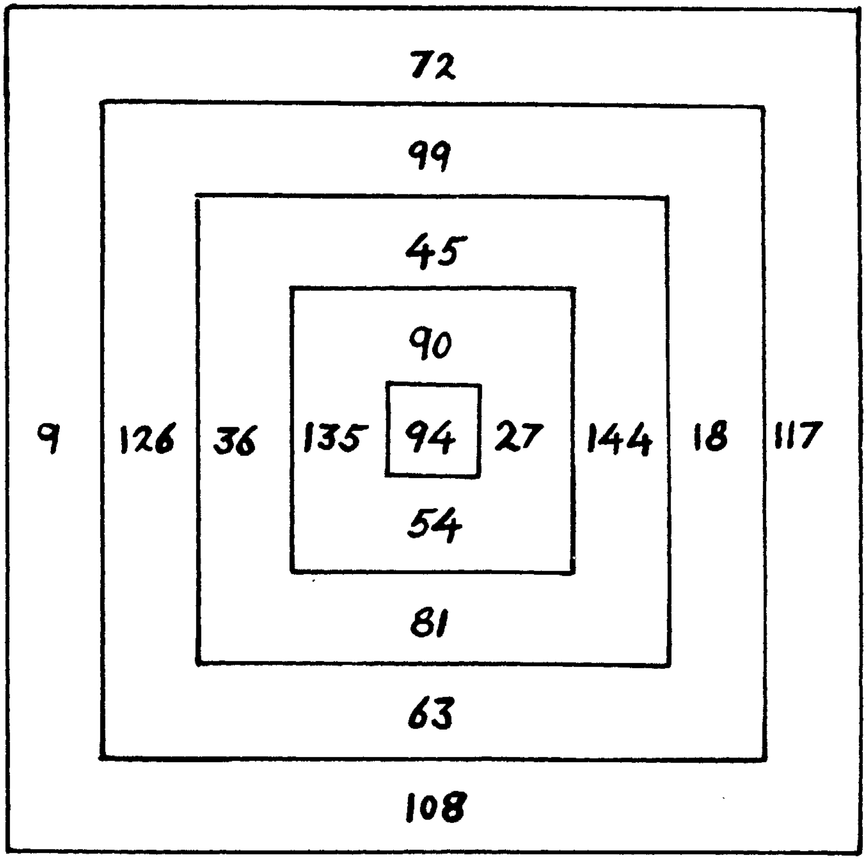
The square is obtained by multiplying each of the numbers of a 3×3 square using the natural numbers by 15. It will be further observed that (400 − 225) = 175 is placed in the centre, so that the sum of each of the arms radiating from the centre may be 400.

Magic Cross: The figure of the magic cross given by Nārāyaṇa is shown in Fig. 111.

This cross has been made with the help of the numbers of the 4×4 square given in Fig. 114. 94 has been placed in the centre to give the required total.

Magic Circles: Nārāyaṇa has given a number of magic circles each with total 400. These circles together with their key squares or rectangle are:

(i) *Magic Circle* from a 3×3 square using the series whose first term is 15 and common-difference 15 (Figs. 112 and 113).



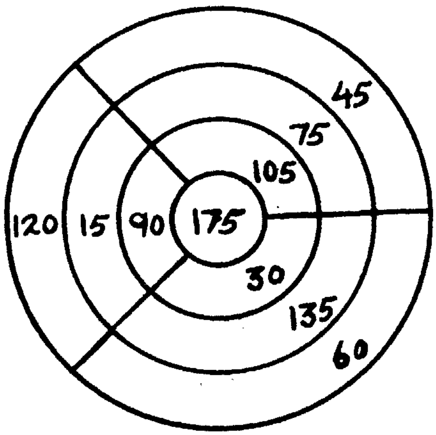
Total = 400

Fig. 111

120	45	60
15	75	135
90	105	30

Total = 225

Fig. 112



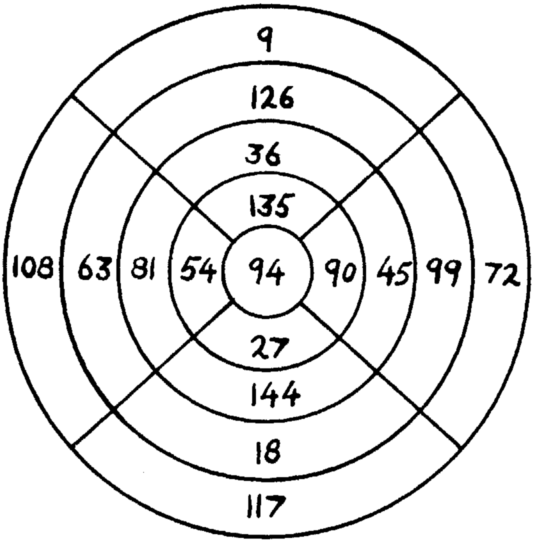
Total = 400

Fig. 113

(ii) *Magic Circle* from a 4×4 square whose first term is 9 and common-difference 9 (Figs. 114 and 115).

9	72	117	108
126	99	18	63
36	45	144	81
135	90	27	54

Total = 306
Fig. 114

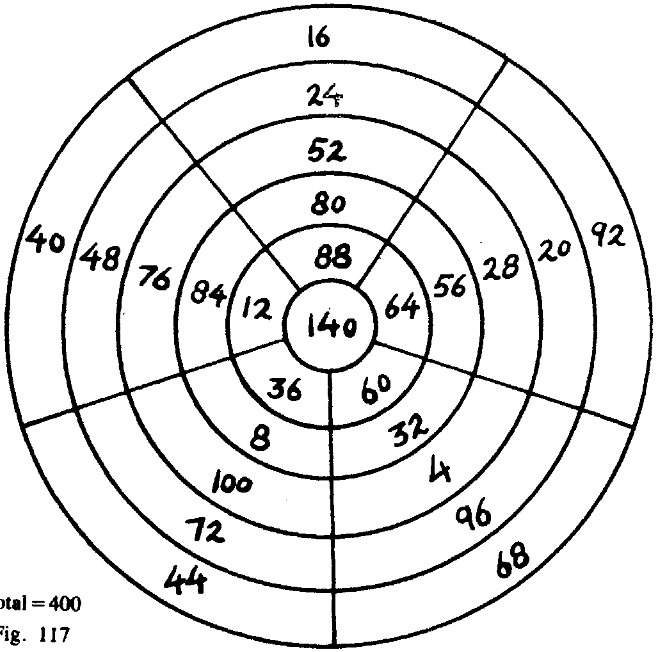


Total = 400
Fig. 115

(iii) *Magic Circle* from a 5×5 square whose first term is 4 and common-difference 4 (Figs. 116 and 117).

68	92	16	40	44
96	20	24	48	72
4	28	52	76	100
32	56	80	84	8
60	64	88	12	36

Total = 260
Fig. 116



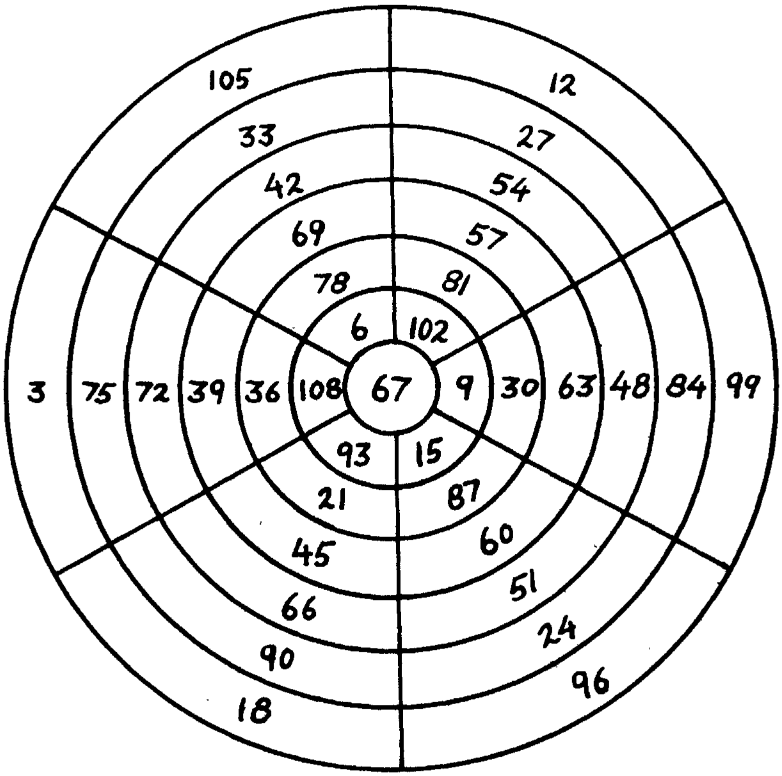
Total = 400
Fig. 117

(iv) *Magic Circle* from a 6×6 square whose first term is 3 and common-difference 3 (Figs. 118 and 119).

3	105	12	99	96	18
75	33	27	84	24	90
72	42	54	48	51	66
39	69	57	63	60	45
36	78	81	30	87	21
108	6	102	9	15	93

Total = 333

Fig. 118



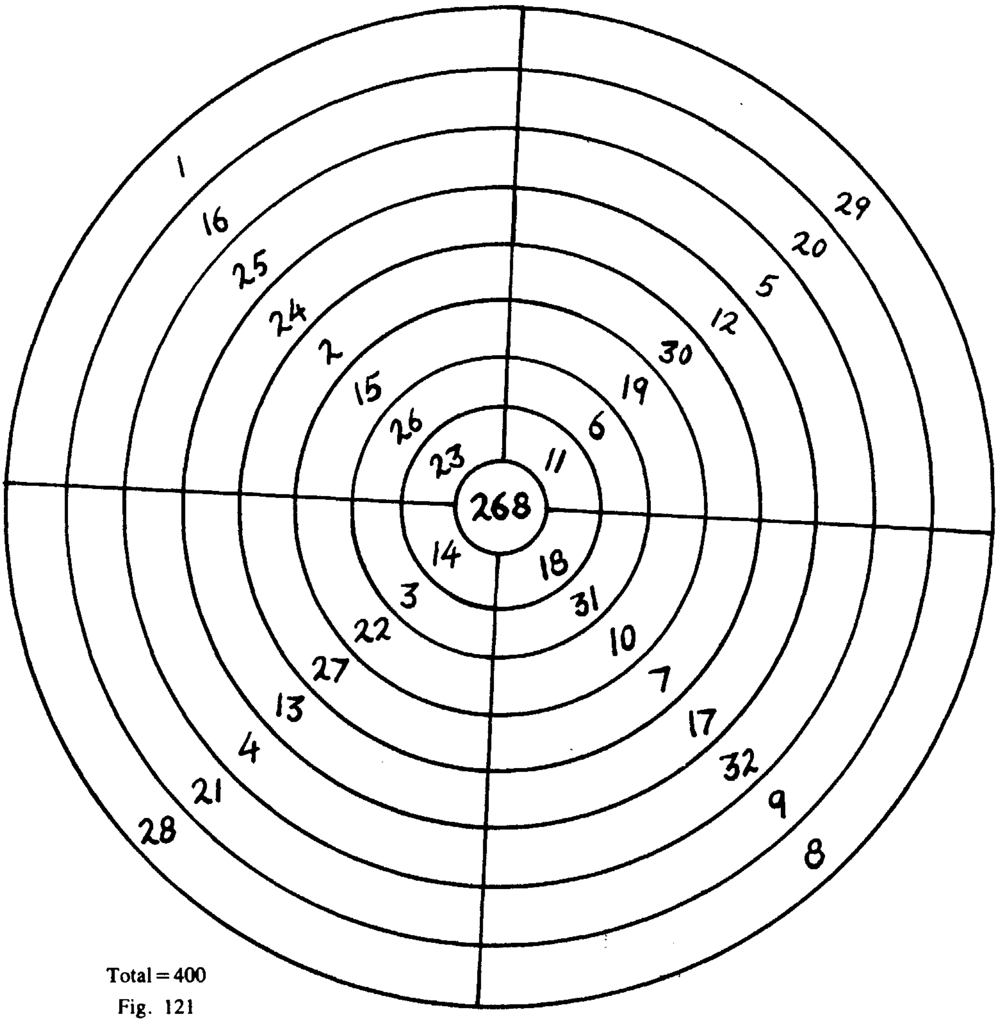
Total = 400

Fig. 119

(v) *Magic Circle* from a 8×4 rectangle using the natural numbers 1 to 32 (Figs. 120 and 121).

1	16	25	24	2	15	26	23
28	21	4	13	27	22	3	14
8	9	32	17	7	10	31	18
29	20	5	12	30	19	6	11

Total rows 132, columns 66
Fig. 120



Total = 400
Fig. 121

Dharmanandana Square: Dharmanandana, a Jaina scholar (circa fifteenth century) has given[47] the following 8 x8 square[48] with total 260 (Fig. 122).

8	7	59	60	61	62	2	1
16	15	51	52	53	54	10	9
41	42	22	21	20	19	47	48
33	34	30	29	28	27	39	40
25	26	38	37	36	35	31	32
17	18	46	45	44	43	23	24
56	55	11	12	13	14	50	49
64	63	3	4	5	6	58	57

Fig. 122

The above square has been constructed by placing the natural numbers 1 to 64 in a 8×8 suqare in the direct order and then shifting the numbers so placed suitably. The square is divided into smaller squares of four cells each. The numbers in those squares that lie on the diagonals are unchanged, while those in the other squares are interchanged with the diagonally opposite ones. The manner of the change will be evident from the key square in Fig. 123 in which the smaller squares that are not to be interchanged are marked by thick letters and thick boundaries.

8	7	6	5	4	3	2	1
16	15	14	13	12	11	10	9
24	23	22	21	20	19	18	17
32	31	30	29	28	27	26	25
40	39	38	37	36	35	34	33
48	47	46	45	44	43	42	41
56	55	54	53	52	51	50	49
64	63	62	61	60	59	58	57

Fig. 123

Dharmanandana's method is quite general[49]. For instance, the 12 × 12 square shown in Fig.125 can be made by dividing the key square (Fig.124) into smaller squares of nine cells.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104	105	106	107	108
109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132
133	134	135	136	137	138	139	140	141	142	143	144

Key-square

Fig. 124

The 4×4 magic square (Fig. 126) based on Dharmanandana's method is interesting as it is not included in Nārāyaṇa's squares (Fig. 36).

1	2	3	141	140	139	138	137	136	10	11	12
13	14	15	129	128	127	126	125	124	22	23	24
25	26	27	117	116	115	114	113	112	34	35	36
108	107	106	40	41	42	43	44	45	99	98	97
96	95	94	52	53	54	55	56	57	87	86	85
84	83	82	64	65	66	67	68	69	75	74	73
72	71	70	76	77	78	79	80	81	63	62	61
60	59	58	88	89	90	91	92	93	51	50	49
48	47	46	100	101	102	103	104	105	39	38	37
109	110	111	33	32	31	30	29	28	118	119	120
121	122	123	21	20	19	18	17	16	130	131	132
133	134	135	9	8	7	6	5	4	142	143	144

Total = 870

Fig. 125

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Key-square

Fig. 126(a)

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Total = 34

Fig. 126(b)

Sundarasūri Squares: Another Jaina scholar, Sundarasūri (circa fifteenth century), has given a number of interesting squares which have been constructed by novel methods[50]. An account of these squares is given below.

3 × 3 square: The filling of the 3 × 3 square as per Fig. 127 is according to the traditional Hindu method, already noted in Nārāyaṇa's work.

4 × 4 squares: A 4 × 4 square with any desired even total may be constructed by giving particular values to n in Fig. 128.

4	9	2
3	5	7
8	1	6

Total = 15
Fig. 127

n−8	n−1	2	7
6	3	n−4	n−5
n−2	n−7	8	1
4	5	n−6	n−3

Total = 2n
Fig. 128

Sundarasūri exhibits the instance with total 32 as per Fig. 129.

In Fig. 129, the number 8 occurs twice, because the total is less than 34, which is the least total for a 4 × 4 square constructed with the series of natural numbers.

Odd squares: Sundarasūri uses the elongated knight's move to obtain the 5 × 5 square shown in Fig. 130.

8	15	2	7
6	3	12	11
14	9	8	1
4	5	10	13

Total = 32
Fig. 129

22	3	9	15	16
14	20	21	2	8
1	7	13	19	25
18	24	5	6	12
10	11	17	23	4

Total = 65
Fig. 130

The method of filling is: Put 1 in the extreme cell of the middle row; move two cells in front and one cell diagonally, and put down the next number 2 and so on. When a block occurs, put the next number in the adjoining cell in the direction of the move, and continue as before[51].

The method can be easily generalised and is applicable to all odd squares. For filling up a $(2n+1) \times (2n+1)$ square the move to be used is n cells horizontally or vertically and one cell diagonally. When a block occurs, the next number is to be put down in front of the cell last filled in the direction of the move. By proceeding in this way, we obtain the required magic square. As an example, we give the 7×7 square as per Fig. 131.

20	28	29	37	45	4	12
44	3	11	19	27	35	36
26	34	42	43	2	10	18
1	9	17	25	33	41	49
32	40	48	7	9	16	24
14	15	23	31	39	47	6
38	46	5	13	21	22	30

Total = 175

Fig. 131

8×8 square: Sundarasūri gives the 8×8 square [52] shown in Fig. 132.

It has been constructed by dividing symmetrically the following key-square into groups of four and two cells. The numbers that lie in groups standing on the diagonals remain unchanged, while those in the others are interchanged with the diagonally opposite ones. The method of division will be apparent from Fig. 133 of the key-square:

1	63	62	4	5	59	58	8
56	10	11	53	52	14	15	49
48	18	19	45	44	22	23	41
25	39	38	28	29	35	34	32
33	31	30	36	37	27	26	40
24	42	43	21	20	46	47	17
16	50	51	13	12	54	55	9
57	7	6	60	61	3	2	64

Total = 260

Fig. 132

Compound magic squares: Sundarasūri gives the 9×9 square shown in Fig. 134.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Key-square

Fig. 133

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

Total = 369

Fig. 134

The method of construction of the above square is apparent from the figure if we consider the square to be divided into nine smaller squares, as is done in the figure given above. It will be found that each of the smaller squares is a 3×3 magic square. Therefore, the method is: Divide the numbers 1-81 into 9 groups in order, and with these groups construct nine 3×3 squares. These nine squares, being numbered one to nine in order, are filled in the bigger square just as in the method of filling a 3×3 square with the numbers 1-9[53].

CONCLUDING REMARKS

The foregoing pages would have shown to the reader that the Hindu achievements in the theory and construction of magic squares stand unsurpassed even up to the present day. The simplest square presenting any difficulty is the 4×4 square whose study began in India as early as the beginning of the Christian Era. The success obtained in constructing this square must have encouraged the consideration of larger squares. The construction of magic squares was not made a part of mathematics, as no theoretical treatment could be given in the earlier stages. There are, however, stray examples of the occurrence of magic squares from the beginning of the Christian Era right up to the time of Nārāyaṇa (1356)[54]. A very elegant and satisfactory method for the construction of the 4×4 square was developed before the time of Nārāyaṇa. This method, which we may call the method of the knight's move, gives us 384 magic squares, which are perfect and possess the characteristics of what are now called "Nasik squares". This method of construction will be new to western scholars of today.

Nārāyaṇa (1356), who undertook the study of these squares, obtained results which have been only recently found in the west by the efforts of several workers. Of his theoretical results, the most important is the demonstration of the fact that magic squares may be constructed with as many series or groups of numbers in A.P. as there are cells in a column. This result was first stated in the west by L.S. Frierson in the beginning of the present century[55]. Another very important feature of Nārāyaṇa's work is the division of magic squares into three types. In our opinion, the recent work done in the west suffers from considerable inelegance because of the absence of such classification.

Nārāyaṇa claims as his own the methods for the construction of $4n \times 4n$ squares and odd squares by means of superposition, and also a method for the construction of $(4n+2) \times (4n+2)$ square. Methods for the construction of certain squares by means of superposition were devised by M de la Hir^ (1705)[56]. Nārāyaṇa's methods, given more than six centuries earlier, are more elegant and practical, although theoretically there is little difference between the two. Nārāyaṇa's method for the construction of $(4n + 2) \times (4n + 2)$ squares seems to be the only general method for the construction of such squares known up to the present.

The squares given by the Jaina monks Dharmanandana and Sundarasūri have evidently been obtained by generalisation of Nārāyaṇa's methods and show that the study of magic squares engaged the attention of the Hindus up to the fifteenth century.

The history of the development of magic squares in India, detailed in the preceding pages, leads irresistibly to the conclusion that the magic square originated in India. The knowledge of these squares might have gone outside India at any time between the first century and the tenth century AD. But it appears to be most probable that the west as well as China got the magic squares from India through the Arabs about the tenth century. This would account for the simultaneous occurrence of the magic square in such far off places as China, Arabia and Western Europe.

NOTES & REMARKS

1. The *Loh Shu* and the map of the Ho are illustrated in *Magic Squares and Cubes* by W.S. Andrews, Chicago, 1908, p. 122.
2. Cf. W.S. Andrews, l.c., p. 123.
3. In a work of Rabbi ben Ezra (c. 1140); cf. D.E. Smith, *History of Mathematics*, II, New York (1923), p. 596.
4. in the work of the Arab philosopher Gazzali, *cf.* Smith, D.E., l.c., p. 597.

5. It is said in the *Vedas* that the gods Indra and Viṣṇu divided 1000 into three. This incident is related in many works.(*Taittirīya Saṃhitā* vii. 1.6.; iii.2. 11; *Atharva-veda*, vii. 44.1.; *Taittirīya Brāhmaṇa*, i.1.6.1; *Śatapatha Brāhamaṇa* iii, 8.4.4. etc.). In the *Taittirīya Saṃhitā*, we have

   "Ye twain have conquered; ye are not conquered,
   Neither of the two of them hath been defeated;
   Indra and Viṣṇu when contended,
   Ye did divide the thousand into three."

   (Keith)

   "The thousand is divided into three at the three-night festival; verily he makes her possessed of a thousand, he makes her the measure of a thousand."

   In the above passages it is not clear what "dividing a thousand into three" means. As the problem was considered so difficult that only the gods could solve it, so it is certain that "division into three" did not mean division into three equal parts or into any three parts or into three parts in arithmetic progression, for division as above can be easily made by the use of ordinary fractions which were known in those times. The passage very probably refers to the construction of a magic square with 1000 as total, especially as it has been stated that it confers benefits acting as a charm if the operation is performed at the three-night festival.

   But to produce this passage as an evidence of the existence of magic squares, without other corroborative facts would, in our opinion, be as unjustifiable as the use of the *Loh-Shu* to establish the existence of the magic square in China in 2200 BC.
6. See *Indian Antiquary*, XI, 1882, pp. 83f.
7. Andrews. W.S., *Magic Squares and Cubes*, Chicago, 1908, p, 125f.
8. To fill this square the mnemonic formula stated by Nāgārjuna is:

   Nīlaṃ^{30} cāpi^{16} dayā^{18}-calo^{36} naṭa^{10}-bhuvaṃ^{44} Khārī^{22}-varaṁ^{24} rāginam^{32}|

   Bhūpo^{14} nāri^{20} vago^{34} jarā^{28} cara^{26}-nibhaṃ^{40} tānaṃ^{06} śataṃ^{100} yojayet||
9. *Bṛhat Saṃhitā* lxxvii. 23 ff.
10. The square as it actually occurs is interspersed with the *bīja* ("elements of a *mantra*").
11. Henceforth we shall translate this term by "number of the square".
12. *Gaṇita-Kaumudī* xiv 4. All the references that follow are from chapter xiv. To avoid unnecessary repetition, the number of the rules only, as found in Padmakara Dvivedi's edition of the *Gaṇita Kaumudī* will be given.
13. This seems to be the first statement of the result that magic squares are made from numbers in A.P., a result on which the whole theory of magic squares is founded.
14. Rule 5, the technical terms are: *mukha* = initial term and *pracaya* = common-difference.
15. Rule 6.
16. Rule 7.
17. Rule 8. The problem is indeterminate. The initial term is arbitrarily assumed, and the common-difference is obtained from the equation $s - n(n-1)d/2 = na$, where $s =$ the given sum, $n =$ the number of cells (or terms), $a =$ initial term, and $d =$ the common-difference.
18. *Caturbhadra* ("four magic square" or "4×4 magic square").
19. Rules 10-12. Nārāyaṇa ascribes the above method to previous writers. It cannot be said how old it is. The squares formed by the method are very popular among Hindu astrologers.
20. Rule 13 (a)
21. Rule 13 (b) – 14(a)
22. Ex. 4.
23. Rule 14(b)-15.
24. Examples 5. These are the first examples of squares constructed by a set of numbers not in a regular A.P.
25. Rules 16-20(a). If the number of *caraṇas* ("rows") be n, and if the first term be assumed to be a and the common difference d, the sum of n^2 terms divided by n, the number of rows, is the total

(*mukhaphala*) of the $n \times n$ square that will be constructed with this series. If the terms are written in rows of n, the initial terms of the rows, i.e., the *mukhapankti*, will be

a, (a + nd), (a + 2nd) ..., [a + n(n − 1) d]

Let the given total be T. The total corresponding to the *mukhapankti* (i.e., *mukhaphala*) is

$$\frac{n^2\left\{a + \frac{(n^2-1)d}{2}\right\}}{n} = \frac{n}{2}\,[2a + (n-1)(n+1)\,d]$$

$$= \frac{n}{2}\left\{a + n(n-1)\,d] + a + (n-1)\,d\right\}$$

which is the form in which the total is expressed by Nārāyaṇa.

$$\textit{Kṣepaphala} = T - \frac{n}{2}\;[a + n(n-1)\,d] + a + (n-1)d\,\}$$

$$= K, \text{ (say)}.$$

We have now to find an arithmetic series of n terms whose sum is equal to K. The terms of this series are added to the corresponding terms of the *mukhapankti*. The rationale of the above result can be easily worked out. It can be easily seen that if A, D are the first term and the common-difference of the series whose sum is K, then the initial terms of the *caraṇas ("rows")* are

[a + A], [(a + nd) + A + D],....., [(a + n(n − 1)d) + A + (n − 1)D]

26. Rules 20(b)-23(a).
27. Here, the term *gaccha* means the "number" of different sets of series that may be obtained for the filling of the square with the required total.
28. Nārāyaṇa gives these initial terms only.
29. Rule 23(b)-24(a).
30. The "number" of the square is the number of cells in a row of the square.
31. i.e., the upper horizontal half of the first square is filled first and then the lower half, and in the second square, the left vertical half is filled first.
32. Rules 24(b)-29. The author Nārāyaṇa was the son of Nṛsiṃha or Nṛhari.
33. It it convenient to take the *parāpankti* such that its sum is less than that of the *mūlapankti*, but this is not essential.
34. In (A′B′), the numbers are repeated. This is due to the fact that the *mūlapankti* and the *parapankti* in this case are the same.
35. This square is practically the same as Frost's "Nasik Square". (W.S. Andrews, l.c., p. 175, Fig. 288).
36. Rules 30-31.
37. It will be observed that all groups of 4 cells have the same total, except the groups included within the thick lines. If we interchange the third and fourth 4×4 squares, we get a 8×8 square in which all groups of 4 cells excepting the centre group have the total 130.
38. The *śliṣṭa* cells are cells not belonging to the diagonal and lying in the two vertical halves of the square. These cells are counted from the boundary inwards as will appear from the examples given. The number of such cells in a $(4n+2) \times (4n+2)$ square is n cells on the right and n cells on the left of each row.
39. Rules 32-36.
40. Rules 37-39.
41. Rules 41-42.
42. Rules 43-45.
43. See W.S. Andrews, l.c., p.1 ff.
44. Rules 46-49.
45. Rules 30-31.
46. i.e. any two lines of numbers that are side by side.

47. The square occurs in the *catuhṣaṣṭi-yoginī-maṇḍala-stuti* of Dharmanandana.
48. This square is given in W.S. Andrews' book, l.c., Fig. 94, p. 43.
49. It is equally applicable to the smaller 4 × 4 square.
50. These squares occur in a *Stotra* by Sundarasūri.
51. W.S. Andrews gives the above method (p.4, Fig. 5) and claims it as his own. He has been anticipated by Sundarasūri by several centuries.
52. The same square has been given by W.S. Andrews, l.c., Fig. 53, p. 25. The square is perfect in all its characteristics. Sundarasūri's method can be generalised to obtain other squares.
53. The above method is now attributed to Prof. Hermann Schubert (cf.W.S. Andrews, l.c., Fig. 96, p. 44). In India it was known several centuries earlier.
54. Magic squares were used as charms and the method of construction seems to have been kept secret by the astrologers who used them in their trade. Another reason for their not occurring more frequently is that they did not belong to any particular subject and so had no place in the literature of the land.
55. Andrews, W.S., l.c., pp. 62 and 151-152.
56. *Memoires de l' Académie Royale* (1705). For a description of the method see also W.S. Andrews, l.c.