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NĀRĀYAṆA'S TREATMENT OF MAGIC SQUARES

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Last chapter of Nārāyaṇa's *Gaṇita Kaumudī* (A.D. 1356) contains a detailed mathematical treatment of magic squares which is unique in several respects.

S. Cammann has dealt with some of Nārāyaṇa's concepts scantily. Religion being his main pre-occupation, mathematical intricacies of Nārāyaṇa's treatment has been left untouched and at times, treated in a manner which the latter did not intend. The present paper deals with the mathematical intricacies of Nārāyaṇa's treatment of the topic.

Nārāyaṇa divides magic squares into three categories, *viz.*, *samabhadra*, *viṣamabhadra* and *viṣama*. He establishes relations between magic squares and arithmetic series. He gives rules for finding 'the horizontal difference' and the first term of a magic square whose total, and total of a row are given. He also gives rules for finding 'the vertical difference' in such a case.

With such mathematical tools in hand and relations established above, Nārāyaṇa gives rules for the formation of all types of magic squares.

INTRODUCTION

Nārāyaṇa wrote his *Gaṇita Kaumudī*[1] in 1356 A.D. Last chapter of the book contains an exhaustive treatment of magic squares and magic figures. He divides[2] magic squares into three categories, *viz.*,

(i) *samabhadra i.e.* double-even magic square,

(ii) *viṣamabhadra, i.e.*, single-even magic sauare,

and (iii) *viṣama, i.e.*, odd magic square.

A magic square having N number of cells on a side is[3] a magic square of order N. Obviously, such a square has N rows, N columns and N^2 cells.

According to Nārāyaṇa the *bhadrāṅka, i.e.*, the order of a magic square is divided by 4. If there be no remainder, the square is[4] a *samabhadra* one (*i.e.*, a double-even magic square). If the remainder be 2, the square is a *viṣamabhadra* one (*i.e.*, a single-even magic square), and if the remainder be 3 or 1, the square is a *viṣama* one (*i.e.*, an odd magic square). Other figures including those in which the number of rows is not equal to the number of columns have been named *upbhadra* by him.

According to Nārāyaṇa, mathematics of all types of magic squares happen[5] to be like the mathematics of series. The (total) number of cells happens to be the number of terms of the series and the square-root of the former happens to be the *caraṇa* of the square.

In a magic square (whose first term is one and common difference is one) the number of terms is added to its square. Half of the sum is equal[6] to the total of the square (which is the sum of numbers in all the cells of the square). The total divided by the square root of the number of terms equals[7] the square's constant (which is the constant to which the numbers in each row, each column, and each main diagonal sum up). Thus, if n be the number of terms (of the series), total of the square $= \frac{n^2+n}{2}$ and the square's constant $= \frac{n^2+n}{2n^{1/2}}$

RULE FOR FINDING THE FIRST TERM AND THE HORIZONTAL DIFFERENCE OF THE MAGIC SQUARE WHOSE TOTAL OF A ROW (*i.e.*, THE SQUARE'S CONSTANT) AND THE NUMBER OF TERMS ARE SEPARATELY GIVEN

Let a be the first term, d the common difference, n the number of terms, and S the sum of an arithmetic series. Also, let s be the sum of the first $(n-1)$ natural numbers. Then,

$$S = \frac{n}{2}[2a+(n-1)d]$$

$$= na+\frac{n(n-1)d}{2}$$

$$= na+sd$$

$$i.e., \; -sd+S = na.$$

Now, if S, s, and n are known the values of a and d can be obtained by solving the above indeterminate equation. In the corresponding magic square a is the first term, and d 'the horizontal difference' which is the constant difference between the numbers in each group of N cells,[8] N being the order of the square.

Nārāyaṇa broke the traditional method of taking 1 as the first term and 1 as the common difference for a magic square and gave rules for the formation of magic squares with numbers different from unity as the first term and 'the horizontal difference'. He says :

> "Considering the negative of the sum of first natural numbers, their number being equal to 'the number of terms (of an arithmetic series) less one', as the dividend, the number of terms as the divisor and the sum as the additive, the quotient and the multiplier together with their respective additives, obtained by the method of pulverisor, happen to be the first term and the common difference (of the series)."[9]

Nārāyaṇa illustrates his rule[10] with the help of the following examples.

By taking $S = 400$ and $n = 16$ and so $s = 120$, the indeterminate equation is obtained as $-15d+50 = 2a$. Solving this equation, the values for a are obtained as 25, 10, -5, *etc.* with 15 as the subtractive (or additive) and those for d are obtained as 0, 2, 4, *etc.*, respectively, with 2 as the additive (or subtractive).

Similarly, by taking $S = 1296$ and $n = 36$ and so $s = 630$, the equation formed is $-35d+72 = 2a$. From this equation, the values for a are obtained as 1, -34, *etc.* with 35 as the subtractive and those for d are obtained as 2, 4, *etc.*, respectively, with 2 as the additive.

Again, by taking $S = 180$ and $n = 9$ and so $s = 36$, the equation obtained is $-4d+20 = a$. From this, the values for a are obtained as 20, 16, 12, 8, 4, 0, *etc.* with 4 as the subtractive and those for d are obtained as 0, 1, 2, 3, 4, 5, *etc.*, respectively, with 1 as the additive.

Finally he observes :

> "Whereever the first term and the common difference are to be obtained there (these should be obtained) from the pulverisor".[11]

Nārāyaṇa's Rule for finding the Vertical Difference

A magic square may be constructed with any number, say a, in the starting position, and with the numbers occupying the first N group of cells, the second N group of cells, ..., and the last N group of cells filled with numbers which vary by some constant difference. This constant difference, say d, between the numbers in each group of N cells is called 'the horizontal difference', as seen earlier. The numbers in any group of such N cells have been named *caraṇa* by Nārāyaṇa.

The difference between the numbers occupying cells N and $N+1$, cells $2N$ and $2N+1$, ..., cells (N^2-N) and (N^2-N+1) may also differ by some constant amount. This constant difference (D) between the numbers occupying cells N and $N+1$, cells $2N$ and $2N+1$, ..., cells (N^2-N) and (N^2-N+1) is called 'the vertical difference'.[12] Nārāyaṇa's rule for finding 'the vertical difference' is as follows.

Let an arithmetic series called *mukha-paṅkti* be a, $a+Nd$, $a+2Nd$, ..., $a+(N-1)Nd$ and another arithmetic series be b, $b+D$, $b+2D$, ..., $b+(N-1)D$ and let the sums of the corresponding terms of these two series be the numbers occupying cells 1, $N+1$, $2N+1$, ..., $(N-1)N+1$. If S be the total of the square,

$$S = (\underline{a+b})+(a+b+d)+(a+b+2d)+\ldots+(a+b+\overline{N-1}d)+(\underline{a+Nd+b+D})+$$
$$(a+Nd+b+D+d)+(a+Nd+b+D+2d)+\ldots+(a+Nd+b+d+\overline{N-1}d)+$$
$$(\underline{a+2Nd+b+2D})+(a+2Nd+b+2D+d)+(a+2Nd+b+2D+2d)+\ldots+\ldots+\ldots$$
$$+(\underline{a+\overline{N-1}Nd+b+\overline{N-1}D})+(a+\overline{N-1}Nd+b+\overline{N-1}D+d)+(a+\overline{N-1}Nd+$$
$$b+\overline{N-1}D+2d)+\ldots+(a+\overline{N-1}Nd+b+\overline{N-1}D+\overline{N-1}d)$$
$$= N^2(a+b)+\frac{(N+1)(N-1)\ N^2d}{2}+\frac{N^2(N-1)D}{2}$$

$$\therefore\quad \frac{S}{N}-\frac{N}{2}\,[2a+(N+1)(N-1)d] = Nb+\frac{N(N-1)D}{2}$$

or $t-m = Nb+\dfrac{N(N-1)D}{2}$, if $t = \dfrac{S}{N}$ and $m = \dfrac{N}{2}\,[2a+(N+1)\,(N-1)\,d]$

or $A = Nb+\dfrac{N(N-1)D}{2}$ where $A = t-m$.

A is called the additive and m, the *mukhaphala*. Nārāyaṇa says :

"The *mukhaphala* subtracted from 'the desired total of a row' happens to be the additive A. From the additive and 'the number of terms' equal to the *caraṇa*, N, the first term b and the common difference D are obtained. The numbers (in the series obtained from them) added to (the respective numbers in) the *mukha-paṅkti* happen to be the first terms of (different) *caraṇas* in all (types of) magic squares".[13]

He illustrates his rule with the help of the following examples.[14]

Let the square's constant be 40. By taking $a = 1 = d$ (and $N = 4$) the *mukha-paṅkti* is obtained as 1, 5, 9, 13.

$$\text{Also,}\quad m = \frac{N}{2}\,[2a+(N+1)\,(N-1)d]$$
$$= 34,$$
$$\text{and so,}\quad A = t-m$$
$$= 40-34$$
$$= 6.$$

Therefore, the equation is obtained as $3 = 2b+3D$.

From this equation the values obtained for b are 0, -3, *etc.* with 3 as subtractive and those for D are 1, 3, *etc.*, respectively, with 2 as additive, With $b = 0$ and $D = 1$, the series is obtained as 0, 1, 2, 3 and with $b = -3$ and $D = 3$ the same is obtained as -3, 0, 3, 6. Adding the terms of the former series to the respective terms of the *mukha-paṅkti* the first terms of *caraṇas* are obtained as 1, 6, 11, 16. Adding the terms of the latter series to the respective terms of the *mukha-paṅkti* the first terms of *caraṇas* are obtained as -2, 5, 12, 19.

In the other case, let the square's constant be 64. As in the above case, the *mukha-paṅkti* is obtained as 1, 5, 9, 13 and $m = 34$. So $A = 30$ and therefore the equation is obtained as $15 = 2b+3D$. From this, the values obtained for b are 6, 3, 0, *etc.* with 3 as subtractive and those for D are 1, 3, 5, *etc.*, respectively, with 2 as additive. With $b = 6$ and $D = 1$, the series obtained is 6, 7, 8, 9 ; with $b = 3$ and $D = 3$, the series is 3, 6, 9, 12 and with $b = 0$ and $D = 5$, it is 0, 5, 10, 15.

As in the above case adding the respective terms of the *mukha-paṅkti* to those of the above series, one by one, the first terms of *caraṇas* are obtained as 7, 12, 17, 22 from the 1st series, 4, 11, 18, 25 from the 2nd and 1, 10, 19, 28 from the 3rd.

Nārāyaṇa's rules for the formation of magic squares utilizing a 'vertical difference' is a mathematical device for obtaining such a square having a given number as the square's constant. Clearly, he has not tried to render the methods look complex to conceal the relatively easy mode of construction as Cammann thinks.[15]

Alternative Rule for finding the Vertical Difference for Even Magic Squares

Let S be the sum of an arithmetic series a, $a+d$, $a+2d$, ..., its number of terms being N^2. If 0 be the additive at $\frac{N^2}{2}$ places and $\frac{A}{N/2}$ at another $\frac{N^2}{2}$ places, the sum of the resulting series

$$= \frac{N^2}{2}\,[2a+(N^2-1)d]+\frac{A}{N/2}\cdot\frac{N^2}{2}$$

$$= \frac{N^2}{2}\,[2a+(N^2-1)d]+A.N$$

$$= \frac{N^2}{2}\,[2a+(N^2-1)d]+\left\{\frac{S}{N}-\frac{N}{2}\,[2a+(N^2-1)d]\right\}N$$

$$\text{as } A = \frac{S}{N}-\frac{N}{2}\,[2a+(N^2-1)d]$$

$$= S$$

Also, any integer added to each of the numbers at $\frac{N^2}{2}$ places and the same integer subtracted from each of the numbers at $\frac{N^2}{2}$ places would not affect the total. So, the other set of additives for the same result will be 1, $\frac{A}{N/2}-1$; 2, $\frac{A}{N/2}-2$; 3, $\frac{A}{N/2}-3$; .... The first number in each set will be the additive at $\frac{N^2}{2}$ places and the second number, a subtractive at the other $\frac{N^2}{2}$ places.

Also, the 1st terms of *caraṇas* will be obtained by adding the first member of a set to each number in one half of the *mukha-paṅkti* and by subtracting the second member of the set from each number in the other half.

The number of sets of additives will be $\frac{A}{N}+1$, and $\frac{A}{N}$ will be either integral or divisible by $\frac{N}{2}$. Hence follows the following rule of Nārāyaṇa :

> "The additive is divided by *caraṇa*. 'The quotient added to one' happens to be the number (of formations). The remainder after such division is either half of *caraṇa* or zero. If the remainder is otherwise, the magic square is defective.
>
> "Cipher and half of 'the additive' (for $N = 4$) with plus one and minus one as common differences are the measures of additives.
>
> "Numbers at half of the places of the *mukha-paṅkti* are added (to the first additive) and those at the other half, (to the other) additive, separately. (The results) are the 1st terms of *caraṇas* in the double-even and single-even magic squares".[16]

Nārāyaṇa illustrates[17] his rule with the help of two examples. In both the examples he takes $a = 1 = d$ and $N = 4$ so that the *mukha-paṅki* is 1, 5, 9, 13.

In the first example the square's constant, t, is supposed to be 40. Also,

$$m = \frac{N}{2}[2a+(N+1)(N-1)d]$$

$$= 34.$$

So, the additive $= t-m$

$$= 40-34$$

$$= 6.$$

Therefore, the sets of additives are 0, 3 ; 1, 2 ; *etc.* Adding the first member of the first set, 0, to each member of the first half of the *mukha-paṅkti* and the second member of the set, 3, to each member of the second half of that, the 1st terms of *caraṇas* are obtained as 1, 5, 12, 16. Similarly, from the second set, the 1st terms of *caraṇas* are obtained as 2, 6, 11, 15.

In the second example, the square's constant is supposed to be 64. So, the additive is obtained as 30. Therefore, the sets of additives are obtained as 0, 15 ; 1, 14 ; 2, 13 ; ... ; 7, 8. From these, as in the above case, the 1st terms of *caraṇas* are obtained as 1, 5, 24, 28 ; 2, 6, 23, 27 ; 3, 7, 22, 26 ; ... ; 8, 12, 17, 21; in order.

With these mathematical tools and relations in hand Nārāyaṇa gives rules for the formation of all types of magic squares.

References

[1] *The Gaṇita Kaumudī by Nārāyaṇa Paṇḍita* (*GK*), Ed. by Padmakara Dvivedi, Princess of Wales Sarasvati Bhavana Texts, No. 57, Pt. II, Banaras, 1942; colophon towards the end of the book. The colophon is

गजनगरविमित ।278 शाके दुर्मुखवर्षे च वाहुले मासि ।
धातृतिथौ कृष्णदले गुरौ समाप्तिगतं गणितम् ॥ 5 ॥

[2] *Ibid.*, xiv, 3a–3b. The text is,

समगर्भे विषमगर्भं विषमं चेति त्रिधा भवेद् भद्रम् ।

[3] Fults, J. L., *Magic Squares*, The Open Court Publishing Co., La Salle, Illinois, 1974, p. 6.

[4] *GK*, xiv, 3c–4. The text is,

संकीर्णमन्डले ये ते उपभद्राभिधे स्याताम् ॥ 3 ॥
भद्राङके चतुराप्ते निरग्रके तद् भवेच्च समगर्भम् ।
द्व्यग्रे तु विषमगर्भम् त्र्येकाग्रे केवलं विषमम् ॥ 4 ॥

[5] *GK*, xiv, 5a–5b and 6c–7b. The text is,

सर्वेषां भद्राणां श्रेटीरीत्या भवेद गणितम् ··· ॥ 5 ॥
··· यद्यावन्ति गृहाणि श्रेटी विषय भवेद् गच्छः ॥ 6 ॥
भद्रे कृतिगत कोष्ठे तन्मुलं जायते चरणः ।

[6–7] *GK*, xiv, 8. The text is,

सपदः पदवर्गोऽर्धम् रूपादिचयेन भवति सङ्कलितम् ।
तत् पदमूलेन हृतं फलं भवेदिष्ट भद्रे वै ॥ 8 ॥

[8] Fults, *Magic Squares*, p. 30.

[9] *GK*, xiv, 9. The text is,

व्यक्पदाय: क्षयगो भाज्यो गच्छो हर फलं क्षपः।
कुटटकजौ लब्धिगुणौ सक्षेपौ मुखचयौ स्याताम् ॥ 9 ॥

[10] *GK*, xiv, Ex. 2. The example is,

पुर्वोदितेषु च गृहेषु धनानि विद्वन्,
खाभ्राब्धयोऽङ्गनिधि नेत्र भुवः क्रमेण।
खेभेन्दवः कथय वक्रचयावभिन्नौ
यद्यस्ति ते गणित कोविदताभिमानः ॥ 2 ॥

The earlier example referred to in the above example is (Ex. 1),

षोडशगृहके षट्कृतिगृहके नवके च कथयाशु।
रूपादिरूपदृष्टया पृथक् पृथक किं फलं भवति ॥ 1 ॥

[11] *GK*, p. 358. Nārāyaṇa's comment is,

यत्र यत्रादयुत्तरानयनं तत्र तत्र कुट्टकाज्ज्ञेयम्।

[12] Fults, *Magic Squares*, p. 31.

[13] *GK*, xiv, 18c–20b. The text is,

मुखफलहीनमभीप्सितफलं भवेत् क्षेपफलसंज्ञम् ॥ 18 ॥
क्षेपफलाच्चरणमिते गच्छे च मुखोत्तरौ समुत्पाद्य।
तच्छेदयङ.कान्मुखपङ्क्तयङ्.केषु क्षेपयेत् क्रमेणैव ॥ 19 ॥
चरणादयः स्युरेवं सर्वेषामेव भद्राक्षाम्।

[14] *GK*, xiv, 16–20ff.

[15] Cammann, S., Islamic and Indian Magic Squares, Pt. II, *History of Religions*, **8**, 276 (footnote), 1968-69.

[16] *GK*, xiv, 20c–23b. The text is,

क्षेपफलं चरणहृतं लब्धं सैकं प्रजायते गच्छः ॥ 20 ॥
भागो निरग्रको वा चरणदलसमावशेषको नियतम्।
यद्यन्यथावशेषं तद् भद्रं जायते तु खिलम् ॥ 21 ॥
शून्यक्षेपफलाद्धप्रमितावादी धनर्णरूपचयौ।
मुख पङ्क्तेः पूर्वदलं स्थानेष्वपि परदलषु च क्षेपौ ॥ 22 ॥
एवं चरणाद्याः स्युः समगर्भे विषमगर्भे च।

[17] *GK*, xiv, 20–23ff.