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SOME EQUALIZATION PROBLEMS FROM THE BAKHSHĀLĪ MANUSCRIPT*

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INTRODUCTION

The *Bakhshālī Manuscript* (=*BM*) is the name given to the mathematical work written on birch-bark and found in 1881 near the village Bakhshālī (or Bakhshalai), in the Peshawar district, situated about 70 miles from the famous Taxila (now in Pakistan). It was passed on for study and publication to Dr. Rudolf Hoernle of the Calcutta Madrasa. After working on the manuscript for about 20 years, he presented it to the Bodleian Library, Oxford in 1902 where it is still there (Shelf mark: MS. Sansk. d.14).

The *BM* consists of 70 leaves arranged in the present order by Hoernle from the mass of sheets which reached his hands. The language of the work is Gāthā dialect or the Western Prakrit which was used in the then N. W. India till about 300 A.D. (according to Hoernle).[1] It is an admixture of Sanskrit and Prakrit. The script of the manuscript is the Śāradā which is said to be developed from the Siddha-mātṛkā about the 8th century A.D. in the N. W. Indian subcontinent.[2] According to Datta,[3] the work is not a treatise on mathematics in the true sense but a running commentary on some earlier original text. Thus we must distinguish between

(i) the date of the original treatise consisting of the *sūtras* (rules) and *udāharaṇas* (examples) only,

(ii) the date of the commentary which the present *BM* work is and which consists of rules, examples, solution of the examples, verification, etc. and

(iii) the date of the present copy of the manuscript which may be quite late and involves many scribes.

Datta places the *BM* work in the early centuries of the Christian era while the present manuscript is placed in the 9th century by Hoernle.[4] Anyway, there is a need to make a fresh and thorough study of the *BM* including the arrangement of the text especially in the light of new findings.[5]

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EQUALIZATION PROBLEMS

The basic rule for solving simple equalization problems is given explicitly in the *Āryabhaṭīya,* II, 30 of Āryabhaṭa I (b 476 A.D.) as follows[6]

"Divide the difference between the *rūpakas* (or rupees) with the two persons by the difference between their *gulikās* (precious things, animate or inanimate). The quotient is the value (in terms of rupees) of one *gulikā* if the possessions (of the persons) are equal."

That is, if one person has a *gulikās* (such as horses, gems, etc.) and b rupees, and another has c *gulikās* and d rupees, then for the equality of their wealths

$$ax + b = cx + d, \qquad .. \qquad (1)$$

where x is the value of each *gulikā,* then

$$x = (d - b)/(a - c) \qquad .. \qquad (2)$$

Instead of equalization of wealth, we may equalize the coordinates (or position vectors) of two travellers which will give us, e.g. the time of their meeting. In fact, the very next verse given by Āryabhaṭa is in this connection. If s_1 and s_2 are the initial distances (from origin) of two travellers who start simultaneously and move along a line (passing through the origin) with constant speeds v_1 and v_2, then their meeting time will be given by

$$s_1 \pm v_1 t = s_2 + v_2 t$$

or

$$t = (s_1 - s_2)/(v_2 \pm v_1) \qquad .. \qquad (3)$$

Here $(s_1 - s_2)$ represents their initial distance apart, and $(v_2 \pm v_1)$ is their relative velocity.

We have quoted the *Āryabhaṭīya* because it is the earliest extant work whose *date is certain,* otherwise similar rules and examples are found in the *BM* (which may be earlier (?) than the time of Āryabhaṭa). For example, *BM,* folio 3r, rule 15 (Kaye III, p. 171) states :

Sūtram : gatisyaiva viśeṣañca vibhaktaṃ pūrva gantunāḥ, tenaiva kālaṃ bhavati......

"*Rule:* Divide the distance already covered by the previous traveller by the difference of (their) speeds; that gives the time (of their meeting)...."

That is, $t = s_1/(v_2 - v_1) \qquad .. \qquad (4)$

by taking the origin at the starting point of the other traveller. An example on this rule is found in *BM,* folio 4r (Kaye III, p. 175) which has been restored by Kaye as

'One goes at the rate of 5 *yojanas* for 7 days, and then a second starts at the rate of 9 *yojanas* a day. When will they have travelled equal distance?'

This is fully worked out in the *BM* getting the answer 35/4 days (after the 9 *yojana*-traveller starts). Then the text proceeds to verify this answer by the Rule of Three (*trairāśikena*) but details are missing. Then follows another example with $v_1 = 18$, $v_2 = 25$, $s_1 = 8v_1$ (Kaye III, pp. 175-176).

In (4), s_1 is the initial distance apart which is covered by the relative velocity $(v_2 - v_1)$ to give the time t. Similarly if s_1 is considered the initial stock (*bhāṇḍāgāram*) of any quantity (including wealth), and v_1 and v_2 are rates of earning/income and expenditure/consumption, then t will be the time in which the stock will be fully consumed. The *BM*, folio 60r, rule 52 (Kaye III, p. 216) states this as:

Sūtram: āyavyāya viśeṣaṃ tu vibhajya dṛṣya saṃguṇam/yallabdhoṃ sā bhavet kālam......//

"The known quantity is divided by the difference of earning and expenditure. The quotient becomes the time (when the quantity will be consumed)."

The accompanying example reads (Kaye III, p. 216) :

'In two days one earns five; in three days he consumes nine. His store is 30. In what time will the whole stock be consumed?'

The above principle is applied to another class of equalization (*samadhanā*) problems of which the following is an example (Kaye III, p. 217) :

Udāharaṇa : "In three days one pandit earns a wage of five and a second wise man earns six (*rasa*) in five days. The second is given by the first 7 from his earnings. By this gift, they became equally rich (*samadhanā*). Tell in what time (this happened)?" (*BM*, folio 60v).

By making a gift g, the gap or difference in their wealth will be $2g$ which is to be covered by the difference of the rates of their earnings. So the *BM*, rule 53 for this gives:

"The difference of the daily earnings being divisor of the amount given. Twice the quotient (obtained) is the time when the gift makes their wealth equal." (Kaye, III, p. 216). That is,

$$t = 2g/(e_1 - e_2) \qquad .. \qquad (5)$$

For the above example,

$$t = 14 \Big/ \left(\frac{5}{3} - \frac{6}{5}\right) = 30 \text{ days},$$

as has been worked out (but the *karaṇam* is missing) and verified in the *BM*, folio 61r (Kaye III, p. 217). There is another example of the same type in *BM*, folio 61 (Kaye III, p. 218) :

'Two *rājaputras* are servants of a king. The wages of one are 2 plus 1/6 a day, of second 1 plus 1/2. The second is given 10 *dīnāras* by the first. Calculate and tell me quickly in what time there will be equality *(samatām)*'.

The answer found and verified is 30 days. Still another example ($g=7$, $e_1=7/4$, $e_2=5/6$) is there in *BM*, folio 31r (Kaye III, p. 186).

Samadhanā Problems Involving Simultaneous Equations

The *BM*, folio 3 (Kaye III, pp. 170-171) contains an interesting problem involving mutual gifts by three merchants. The text is not fully preserved but can be restored, from the workingout details given, as follows :

Example : "One possesses 7 horses *(aśvas)*, another 9 *hayas*, and the third 10 camels. Each gives one of his animals to both the others (by which they become equally wealthy). It is required to find the capital of each merchant or the price of each animal. If thou art able, solve me this riddle."

Let x, y, z be the price of a horse, *haya*, and camel respectively, and let the merchants possess a, b, c of these animals (instead of 7, 9, 10). After the mutual gifts are made, let S be the wealth of each merchant. So that we must have

$$(a-2)x + y + z = S,$$
$$x + (b-2)y + z = S,$$
$$x + y + (c-2)z = S. \qquad \ldots \quad (6)$$

These equations can be reduced to the type

$$rx+k = sy+k = tz+k = S, \qquad \ldots \quad (7)$$

where

$$k = x + y + z,$$
$$r = a-3,\ s = b-3,\ t = c-3.$$

Since S is not known, the set (7) is indeterminate and will have infinite solutions given by

$$rx = sy = tz = S-k. \qquad \ldots \quad (8)$$

One set of simple integral solutions is obviously

$$x = st,\ y = tr,\ z = rs \qquad \ldots \quad (9)$$

or

$$x = P/r,\ y = P/s,\ z = P/t, \qquad \ldots \quad (10)$$

where

$$P = S-k = rst. \qquad \ldots \quad (11)$$

The *BM* solution, as is clear from the given working details (Kaye III, p. 171) is equivalent to (10). *BM* first obtains $r = 4$, $s = 6$, and $t = 7$ from given $a = 7$, $b = 9$, and $c = 10$. Then is obtained $P = rst = 168$, whence

$$x = 42,\ y = 28,\ z = 24. \qquad \ldots \quad (12)$$

Finally, the capitals of the merchants are found to be 7×42, 9×28, 10×24, or 294, 252, 240 respectively and the *samadhanā* wealth of each to be 262.

It must be noted that the above solution (10) is not the least or lowest integral for which we should take the L.C.M. of r, s, t, instead of P in (10). Then the solution will be $x = 21$, $y = 14$, $z = 12$, and not (12). However, the *BM* solution (10) is same as said to be given by Śrīdhara (*c.* 800 A.D.) whose rule is found quoted in the *Kriyā-kramakarī* (*c.* 1534), a commentary on the famous *Līlāvatī* (1150 A.D.), as follows.[7]

Puruṣasamāsena hataṃ dātavyaṃ tadviśodhya paṇyebhyaḥ|

Śeṣaṃ parasparahataṃ svaśeṣabhaktaṃ maṇermaulyam||

"Multiply the number (of gems) to be given by the total number of men (taking part in the exchange) and subtract the product from the number (of gems) for sale (owed by each). The continued product of the remainders divided by any one's own remainder gives the price of his gem."

Of course this rule is more general being applicable to the case of n merchants possessing n types of gems (instead of animals) and making mutual gift of g gems (instead of one). Here we shall have the system

$$r_1 x_1 + k = r_2 x_2 + k = \ldots. = r_n x_n + k = S \qquad .. \qquad (13)$$

where $x_1, x_2, \ldots. x_n$ are the prices of the gems and

$$r_1 = a_1 - ng,\ r_2 = a_2 - ng, \ldots.$$

with

$$k = (x_1 + x_2 + \ldots. + x_n)g.$$

It is clear that the lost (or untraced) *BM sūtra* must have been similar to the above rule. In fact, the *BM* phrase *śeṣaṃ paraspara kṛtaṃ guṇita..svaśeṣena tu vibhaktam* is quite comparable to Śrīdhara's phrase in the above *sūtra*. It must also be noted, however, that the above *sūtra* is found (with only slight variation) in the *Gaṇitasāra-saṃgraha* (=*GSS*), VI, 163 of Mahāvīra (850 A.D.).[8] Mahāvīra's example ($n = 3$, $g = 1$, $a_1 = 6$, $a_2 = 7$, $a_3 = 8$) has been also reproduced (along with some other examples of his) in the *Kriyākramakarī* but without mentioning the source (may be because he was a Jaina).[9] I have also not been yet able to locate the quoted Śrīdhara's *sūtra* in any of his extant works.

The *BM*, folios 1—2 (Kaye III, pp. 168-170) contains another *sūtra* (No. 11) with an elaborate example involving similar theory. The text is mutilated and wrongly interpreted by Kaye. Correct interpretation of the example is given by Gurjar[10] but the *BM sūtra* has not been explained so far (which we attempt to do here). The example is about five merchants (with capitals x_1 to x_5, say) who together want to purchase a jewel of price S, say. The given conditions lead to the system

$$\begin{aligned}(x_1/2) + x_2 + x_3 + x_4 + x_5 &= S\\ x_1 + (x_2/3) + x_3 + x_4 + x_5 &= S\\ x_1 + x_2 + (x_3/4) + x_4 + x_5 &= S\\ x_1 + x_2 + x_3 + (x_4/5) + x_5 &= S\\ x_1 + x_2 + x_3 + x_4 + (x_5/6) &= S \quad .. \quad (14)\end{aligned}$$

These easily reduce to the system (13) with $g = 1$, $r_1 = - 1/2$, $r_2 = - 2/3$, etc. which are not positive integers, and so solution (10) is not given. However, the values of the unknowns are inversely proportional to r_1, r_2, etc. So we can take them proportional to 2/1, 3/2, 4/3, 5/4 and 6/5. The *BM* has reduced these to common denominator *(sadṛśaṃ kriyate)* as

120/60, 90/60, 80/60, 75/60, 72/60.

So that 120, 90, 80, 75, 72 are taken to be the values of the unknowns as an integral solution and the price of the jewel is found to be 377 *(eṣa maṇi mūlyam)*. The rest of the working is done to verify all the equations of (14).

Suppose (p/q) is the fractional coefficient of any unknown along the leading diagonal in (14). The value of the corresponding r-coefficient in (13) will be

$$(p/q) - 1 = -(q - p)/q, \quad q > p.$$

Since the value of the corresponding unknown is inversely proportional to r (and the minus sign can be ignored), the prescribed *BM sūtra* (No. 11) rightly asks

'*Aṃśāṃ viśoddhya-cchedebhya kuryāt tatparivartanam*'

That is, "Subtract the numerator-parts (p) from the denominator (q) and invert it (the fraction)".

Then we reduce the resulting fractions to a common denominator etc. to get a suitable integral solution. This explains the *sūtra* and the working out of the example with it. Interestingly there follows another example in three unknowns for which fractional coefficients along the diagonal in (14) are all negative, namely

$$-7/12,\ -3/4 \text{ (with correction)},\ -5/6$$

so that, in this case by the above *sūtra* and theory, the unknowns will be proportional, 12/19, 4/7 and 6/11. The values of the unknowns are then correctly found to be 924, 836, and 798 respectively and that of jewel's price to be 1095 (Kaye III, p. 170)

Equalization of Uniform and Accelerated Growths

Let

$$\begin{aligned}S_1 &= a + (a + d) + (a + 2d) + \ldots \text{to } n \text{ terms, and}\\ S_2 &= b + b + b + \ldots \text{to } n \text{ terms}\end{aligned}$$

represent the accelerated and uniform growths of any types of quantities. For equalization or *samadhanā* type of problems S_1 is equal to S_2.

Hence

$$[(n-1)\ d/2 + a].\ n = b.n \quad .. \quad (15)$$

or

$$(n-1)\ d + 2a = 2b$$

which is comparable to (1) in $(n-1)$, and hence by (2)

$$n = (2b - 2a)/d + 1 \quad .. \quad (16)$$

$$= 2(b-a)/d + 1 = (b-a)/(d/2) + 1. \quad .. \quad (17)$$

For the purpose of ancient rhetoric mathematics, it is often necessary and useful to distinguish between the above three forms for n. The form (16) is contained in the *BM,* folio 8r (Kaye III, p. 172) as :

Sūtram : *Dviguṇaṃ prabhavaṃ śuddhā dviguṇaṃ niyataṃ tathā.*

Uttareṇa bhajeccheṣaṃ labdhaṃ rūpaṃ vinirdiśet.

That is, "Subtract twice the first term from twice the constant rate, and divide the remainder by the common difference; then add one (to get n)."

The accompanying example is somewhat: 'One servant does work at fixed rate equivalent to 10 *māśakam,* and another works 2 units on first day increasing by 3 on each subsequent days. In what time will there be equality of their work?'

So that here $a = 2$, $d = 3$, and $b = 10$. But full working is not available, the answer will be found to be $n = 19/3$.

An interesting point in the working is that $(2b—2a)$ is correctly found to be 16. Then a wrong phrase '*uttarārdhena bhājayet*' (instead of '*uttarena bhājayet*') was quoted and was afterwards cancelled. It may be that the pharse '*uttarārdhena bhājayet*' was quoted from another *sūtra* which gave the last form (17) of the rule for which it is correct. In fact, an example ($a=3$, $d=4$, $b=7$) on *BM,* folio 7v (Kaye III, p.174) has been indeed worked out by this very form of the *sūtra*. Consequently, it is believed that *BM* folio 7v did contain such a *sūtra* which is lost.[11] This folio has another example ($a=1$, $d=2$, $b=5$) which began to be solved by the same *sūtra* but the details are lost.

The example in *BM,* folio 9r has been restored by Hoernle as (Kaye III, p. 173):

"For a certain feast one Brāhmaṇa is invited on the first day, and on every succeeding day one more Brāhmaṇa is invited. For another feast 10 Brāhmaṇas are invited every day. In how many days will their numbers be equal, and how many Brāhmaṇas were invited."

This example ($a=1$, $d=1$, $b=10$) is fully worked out in the text getting $n=19$ days, and then by *'rūpoṇa karaṇa'*, i.e. L.H.S. of eq. (15), the number of Brâhmaṇas invited in the first feast

$$=[(19-1).(1/2)+1].19 = 190$$

which will obviously be also the number for the other feast.

As a word-numeral *rūpa* means one (in algebra *rūpa* stands for the absolute term) and *BM's* frequently used phrase *'rūpoṇa karaṇa'* refers to the formula

$$[(n-1).d/2+a].n = S_n$$

The *GSS*, II, 61 (p. 20) which contains this formula also starts with the phrase *'rūpeṇono'*, This work, VI, 319 (p. 173) contains the last form (17), while the other form in (17) is found in the *Pāṭīgaṇita*, rule 96, of Śrīdhara.[12]

Equality of Two Uniformly Accelerated Growths

Let,

$$S_1 = a + (a+d) + (a+2d) + \ldots\ldots \text{to } n \text{ terms},$$
$$S_2 = b + (b+e) + (b+2e) + \ldots\ldots \text{to } n \text{ terms},$$

If these two are equal, we must have

$$(n-1)d + 2a = (n-1)e + 2b$$

which is again comparable to (1), and hence by (2)

$$n = 2(b-a)/(d-e) + 1 \qquad \ldots \quad (18)$$

This formula is contained in *BM*, folio 4v, rule 17 (Kaye III, p. 176) as follows :

Ādyor viśeṣa dviguṇaṃ cayasuddhir-vibhājitam
Rūpādhikaṃ tathā kālaṃ gati sāmyaṃ tadā bhavet.

"Twice the difference of the initial terms divided by the difference of the common differences is increased by one. That will be time (represented by *n*, cf. *kāla iha pada-syopalakṣaṇam*)[13] when the distances moved (by the two travellers) will be same."

The accompanying example reads :

"The initial speed (of a traveller) is 2 and subsequent daily increment is 3. That of another, these are 3 initially and 2 as increment. Find in what time will their distances covered attain equality."

The working is lost, but the answer, by (18),

$$= 2(3-2)/(3-2)+1 = 3 \text{ days}.$$

Another example is found in *BM,* folio 5r (Kaye III, p. 177). It is on attaining *samadhanā* (equality of wealth), the given rates of earning are $a=5$, $d=6$, and $b=10$, $e=3$. The example has been fully worked out by (18) and verified, by the usual *'rūpoṇākaraṇa'*. The answer obtained is n equal to 13/3 (days) when each would have pooled 65 units of wealth. It must be noted that the value of n found is *fractional* which the ancient Indians accepted even as *number of terms* of a series. Of course here there is no difficulty as it represents time.

Now, we discuss the interesting situation presented in one portion of the *BM,* folio 4 (Kaye III, p. 176). The available part of a *sūtra* (No. 16) in the recto side of the folio, and the partially extant working of an example in the beginning of the available verso side shows that the intended rule was the following form of (18):

$$n=[(b-a)/(d-e)]\times 2+1 \quad .. \quad (19)$$

Since the beginning part of the worked out example is not available, we cannot be sure of given data which is only to be guessed or constructed. Here I shall discuss Kaye's conjectures and then give my own restoration with explanation. All that follows from the available working is that, in the said example, difference of first terms $=2$ $=b—a$, say and difference of the common differences $=2=d—e$ say.

Hence by (19), $n=3$. Also then the sum of first series is found to be 21 (which should also be the sum of the other series). So that we must have (assuming no other lacuna) $(7—d)$, 7, $(7+d)$ and $(7—e)$, 7 $(7+e)$ as the terms of the series. Now for finding d and e we get only one relation, namely, $d=2+e$. So that there can be many possibilities even of simple type namely (i) 4, 7, 10 and 6, 7, 8; (ii) 3, 7, 11 and 5, 7, 9; (iii) 2, 7, 12 and 4, 7, 10; (iv) 1, 7, 13 and 3, 7, 11.

Kaye took (i) and says it *was the (lost) example.* He gave no reason (as it was only a guess). At the end of the rule there is a statement that *'sūtre bhrāntimasti'* which means that the commentator or scribe found some confusion or mistake in the rule. Kaye explains this away simply by saying that the above remark is about the wrong numbering of the rule, because the numbering phrase (also) happens to be wrong being *'ṣodaśaṃ sūtram* 17'. Actually the figure should be 16 *(ṣodaśa)* instead of 17, but this may be considered a minor scribal slip.[14]

I think that the remark that 'there is confusion or mistake in the *sūtra*' was made because some different type of trouble was faced while working out the example by applying the rule. I advance the following possibility.

When any new or more general rule is given, it is quite natural to illustrate it with the help of examples already given earlier. Here also it happened so. In order to illustrate that the rule corresponds to (19), the commentator took the same two series which were used to illustrate the rule for (17). These are (see under uniform and accelerated growth):

1st series, $a=3$, $d=4$; 2nd series $a=1$, $d=2$, Now ancient rules are verbal (not symbolic) and working is done rhetorically (and not by substitution in a formula as we do now). The original *sūtra* needed the "difference" of the first terms and of common-differences. These were found to be (3—1) =2, and (4—2)=2 for the above series. Then the first difference is to be divided by the second difference just obtained thereby getting 2/2=1 which is to be doubled and then increased by unity. So that the *BM* solver got $n=3$ finally. He then found the sum of three terms of the first series to be 21. But for the second series the sum of three terms will be 9 (as he might have found) but he knew that it should also be 21. And here is the trouble which he faced.

The *BM* solver might have again checked his working steps, but due to lack of symbology and rhetoric way of working, he could not detect the mistake he committed or the fallacy in which he was trapped. But since both the sums should be same (and not 21 and 9 as he got), he was compelled to state that there is some confusion or error in the rule itself. There is also another remark at the end (but the plate is not clear)

kiṃ prabhūtepi likhite

which might mean

"How the (equal sums) are obtained? Even then I am writing them (what they should be)."

Of course, some emendations may also be suggested to make the matters more clear.[15]

The confusion started with the choice of bad example for illustration here because the *a* and *d* of one series are *both* greater than those of the other (so that their sums will never be equal for positive n). Then the *BM* solver fell prey to verbal "difference" which he took as positive or numerical one, thereby getting wrong $n=3$. The correct value, by (19), will be $n=-1$. And then S_1 and S_2 will be both equal to 1. But such theoretical discussions are difficult at rhetoric stage.

Our above interpretations are not impossible. The resulting restorations, if plausible, will have some historical value.

References and Notes

[1]See *The Bakhshālī Manuscript* ed. by G. R. Kaye, Parts I (Introduction) and II (The Text), Calcutta, 1927, Part III (Text re-arranged), Delhi, 1933; Part I, p. 11. Both the volumes have been recently reprinted, Cosmo Publications, New Delhi, 1981. We shall refer these parts as *Kaye I, II, III*, respectively.

[2]See *J. Ancient Indian Hist.*, Vol. 4 (1970-71), p. 117.

[3]Datta, B., "The Bakhshālī Mathematics", *Bull. Calcutta Math. Soc.*, Vol. 21, (1929), 1-60; pp. 4-6,

[4]*Ibid.*, pp. 3-4, and 57,

[5]Gupta, R. C., "Centenary of the Bakhshālī Manuscript's Discovery", *Gaṇita Bhāratī,* Vol. 3 (1981), pp. 103-105 has a good bibliography to start the study.
[6]The *Āryabhaṭīya* with the commentary of Bhāskara I etc., ed. by K. S. Shukla, New Delhi, 1976, p. 126.
[7]The *Līlāvatī* with the *Kriyākramakarī,* ed. by K. V. Sarma, Hoshiarpur, 1975, p. 227.
[8]The *Gaṇita sārasaṃgraha,* ed. with Hindi translation by L. C. Jain, Sholapur, 1963, p. 134.
[9]*Kriyākramakarī* (ref. 7). pp. 228-230.
[10]Gurjar, L V., *Ancient Indian Mathematics and Vedha,* Poona, 1947, pp. 63-66.
[11]Kaye III, p. 174; and Datta (ref. 3), p. 7.
[12]*Patiganita of Sridharacarya,* ed. by K. S. Shukla, Lucknow, 1959.
[13]*Ibid,* p. 139 (of the Sanskrit Commentary).
[14]Datta (ref. 3), p. 7.
[15](Added in the proof) : such as that suggested by T. Hayashi in his Doctoral Thesis (Brown University, 1985). But his restoration is unsatisfactory. I received his thesis too late to make any use for this paper.