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ON TRISECTION OF AN ANGLE LEADING TO THE DERIVATION OF A CUBIC EQUATION AND COMPUTATION OF VALUE OF SINE

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Many attempts have been made to solve the problem of trisection of an arc or angle since antiquity. In this paper a method suggested by an Indian mathematician Ghulam Husain Jaunpnri (b. 1790) is highlighted. Derivation of a cubic equation by it and its application in computation of the value of sine are also discussed.

INTRODUCTION

The problem of trisection of an angle is classical and in every century it has been exercising the minds of eminent mathematicians. The objective was to calculate sine of the angles for different arcs for computation of two planetary positions, specially for those arcs or angles which could not be taken without trisecting an arc or the angle. In doing so, many attempts had been made by different mathematicians like Archimedes, Nicomachas, Al-Berūnī, Thābīt b Qurra, Pappus, Al-Sijzī, Vieta and an Indian mathematician of early 19th century Ghulām Ḥusain Jaunpūrī (b. 1790) etc. and they applied various methods: Conic sections, transcendental curves, circle and neusis. Some of them reduced the problem to a cubic equation and by its solution computed the value of sine of the arcs.

In this paper the method of trisection of an angle by means of so called neusis method (i.e. insertion of a static line) and derivation of a cubic equation equivalent to this problem and by its solution the computation of value of sine of an arc or angle are discussed.

CONSTRUCTION

Although trisection of an arc or angle by above-mentioned method has been suggested by many mathematicians but none of them explains how the static line is to be inserted, which is, in fact, the basic part of the construction. Keeping this in view we consider first Proposition 8 of the *Book of Lemmas* of Archimedes (287 BC—212 BC) where he states:[1]

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If AB be any chord of a circle whose centre is O and if AB be produced to C so that BC is equal to the radius, if further CO meet the circle in D and be produced to meet the circle a second time in E the arc AE will be equal to three times the arc BD, i.e. $\angle BOD = \frac{1}{3} \angle AOE$. (Fig. 1).

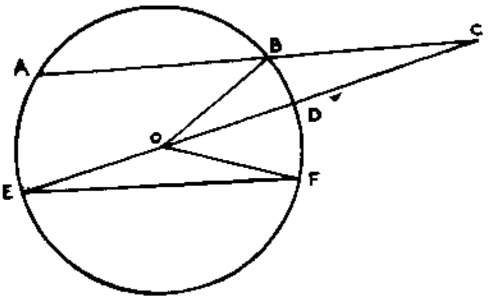

Fig. 1

But in this construction the arc to be trisected is not specified. In fact, first an angle BOD is obtained subsequently leading to arc AE thrice in size of the arc BD. As Smith[2] also points out "this however is manifestly no solution of the problem".

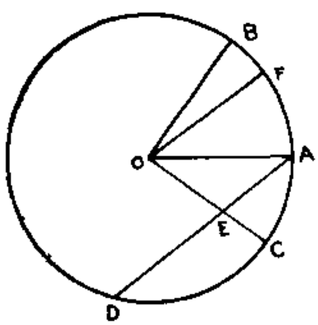

Fig. 2

Companus (b. 1260) at the end of *Book IV* of his translation of the *Elements* describes the trisection of an angle.[3] He considers the angle AOB that is to be trisected be placed with its vertex at the centre of a circle of any radius $OA = OB$. From O draw a radius OC perpendicular to OB and through A place a straight line AED in such a way that $DE = OA$. Finally through O draw line OF parallel to AED. Then $\angle FOB$ is one-third of $\angle AOB$ as required. (Fig. 2).

Here Companus does not specify how the straight line AED will be placed such that $DE = OA$.

Franciscus Vieta (1540-1603) a French mathematician also suggests as follows:[4]

"Let AOB be the angle to be trisected. Describe any circle with centre O. From B draw BRQ so that the part RQ, intercepted between the circle and the diameter AS produced, shall be equal to the radius. Hence $< AOP = \frac{1}{3} \angle AOB$". (Fig. 3).

But we can easily see that Vieta also does not give any method for drawing the line BQ although it is the basic part of the construction.

This problem was also considered by an Indian mathematician and astronomer

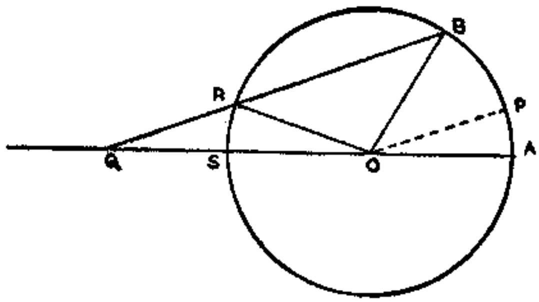

FIG. 3

of late eighteenth century Ghulām Ḥusain Jaunpūrī (b. 1790). He points out in his Compendium *Jāme'-i-Bahādur Khānī*, composed in 1833,[5] that

"It should be understood that sages among the ancients and moderns till today have not succeeded with arguments involving static lines, but the objective is attained by moving the straight line....and its verification will have to be done by a divider such that the distance between the two legs is equal to the radius."

In describing his solution he states:[6]

Suppose the arc *AB* of the circle *ABJD*, whose centre is *E*, is to be trisected. But the arc should not be greater than the quarter. Extend the diameter *AJ* towards *J* up to *Z*, we take a straightedge and one side of which is placed at two points *B*, *J*. The point *B* which corresponds to *J*, is fixed. We move the straightedge from *J* towards *Z*. It is obvious that after a little movement the magnitude of *HJ* in the straightedge which is enclosed between the circumference of the circle and the line *JZ* will be less than the half diameter of the circle and gradually the movement of *HJ* will increase without end. It is inevitable that at some stage it will be equal to the half diameter *JE* and for verification the divider is used; the distance between the two legs will be equal to the radius. So that $HT=JE$. The arc *JH* separates equal to one-third of the arc *AB*. (Fig. 4).

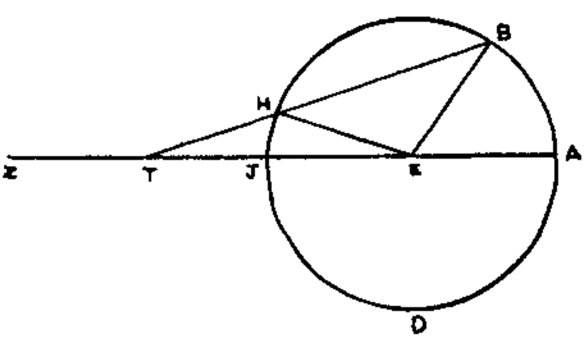

FIG. 4

Thus we see that Ghulām Ḥusain gives a complete treatment of insertion of static line which was not given by any other mathematician before him.

Derivation of a Cubic Equation from Trisection of an Angle

To deduce a cubic equation equivalent to the trisection of an angle,[7] let ch. (α) be the chord of an angle α placed in the centre of a circle of radius 1. The trisection of the angle α is equivalent to the construction of a segment of a straight line of length ch ($\alpha/3$). We have the relation $[\text{ch}(\alpha/3)]^3+\text{ch}(\alpha)=3\,\text{ch}(\alpha/3)$. So trisection of an angle α is equivalent to the solution of the equation

$$X^3+\text{ch}(\alpha)=3X$$

where ch $(\alpha/3)=X$. In special case if

$\alpha=60°$, we have

$$X^3+1=3X.$$

Abul Jūd (10th century) derived the same cubic equation where he considered X as the side of a regular 18 sided polygon in the circle with radius 1 (trisection is the special case of it). Al-Berūni (973-1048) also in his book "*Qanūn al-Masūdi*" derived the same equation but by repeating proposition of Abul-Jūd*. His derivation is as follows:[8]

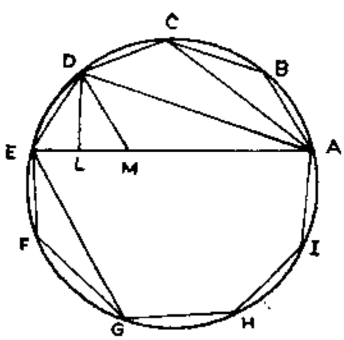

Fig. 5

"The circumference of the circle is divided into 9 equal parts at the points A, B, C, D, E, F, G, H, I, we have the chords AE and EG and the broken line AEG is inscribed in the circle. From the middle point D of the arc ADG let there be a perpendicular DL to AE. (Fig. 5) Then by Archimedes theorem:

$$\frac{AE+EG}{2}=EL+EG$$

from which $EL=\frac{1}{2}(AE-EG)$.

Let $LM=EL$, then $AM=EG$. Notice that $\angle DEL$ is based on the arc AD equal to 120°, from which $\angle DEL=60°$.

Hence equilateral triangle EDM will have equal sides and $DE=EM$.

Let $DE=1$ and $EG=X$. Then $AE.EG=(1+X)X$ and by second theorem on broken line inscribed in the circle:

*J. P. Hojendijk remarks that Abul-Jūd and Al-Berūnī both did not know that the trisection of an arbitrary angle was equivalent to a cubic equation.

$$DE^2+AE.EG=AD^2$$

i.e. $1+(1+X)X=AD^3$. (i)

As the segment ADE is inscribed in the circumference AEC, as per second theorem of broken lines inscribed in circle

$$DC^2+AD.DE=AC^2$$

Since $AC=EC=X$, $DC=DE=1$,
therefore $AD=X^2-1$.

Substituting this in (i) we get

$$1+(1+X)X=(X^2-1)^2$$
$$1+X+X^2=X^4-2X^2+1$$

and after restoration and opposition we get

$$X^3=1+3X.$$

Similarly by means of analogy Berūnī arrives at the equation

$$X^3+1=3X$$

where X is the side of 18-gon inscribed in the circle of radius 1 and he computes the approximate value of X of this equation in sexagesimal fraction,

$$X=20^{I}\ 50^{II}\ 16^{III}1^{IV}$$

where $1^{I}=1/60$, $1^{II}=1/60^2\ldots.)$

and in interpretation of decimal fraction it comes as

$$X=0.3472963722. \qquad \text{(a)}$$

Schoy in his book "*Die Trigonometrischen Lehren*" mentions the exact solution of this equation as[9]

$$X=20^{I}\ 50^{II}\ 16^{III}0^{IV}45^{V}$$

or $X=0.3472963529$. (b)

By comparing these two results (a and b) we can conclude that Al-Berūnī's value varies after seven places of decimal and is greater than the actual value by the fraction of 1.93×10^{-8}. He uses this approximate value in a calculation of the sine of one degree.[10]

Ghulām Ḥusain Jaūnpurī also deduces the same equation by trisecting the angle of 60° (which is the special case of regular nonagon). His derivation is entirely different from Al-Berūnī's and is based on Euclid's proposition. He derives as follows:[11]

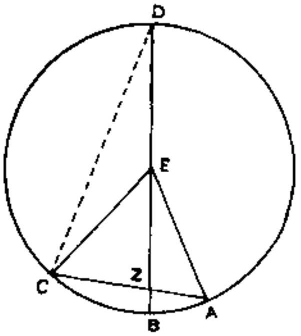

Fig. 6

Suppose the arc ABC is 1/6th of the circumference of the circle. (Fig. 6). Hence its chord will be 60° and the required chord of the arc AB is 20°. Let $AB = AZ^{*} = X$ the *Shai* (thing), hence the *māl* (square) of this will be X^2. Since AZ is mean proportion between the radius AE and BZ, the *māl* will be equal to the surface $AE.BZ$ and when we divide the *māl* by AE which is 60°, the quotient which is one minute of X^2 (i.e. $X^2/60$) will be equal to BZ[6].

Now I say, by proposition 27 of chapter one, section three, (i.e. if two chords intersect each other in the circle, the surface bounded by the segments of one chord will be equal to the surface bounded by the segments of the other), the surface $AZ.ZC$ will be equal to $ZB.ZE$. And since ZC is 60° less *Shai*, i.e. $60-X$, the surface $AZ.ZC = X(60-X)$. And that BZ is $X^2/60$, therefore DZ will be two *murfúa* (elevation) less one minute of X^2,

i.e. $DZ = 120 - X^2/60$.

The surfaee $DZ.ZB$ is two *māl* less one second of *māl* at *māl*,

i.e. $DZ.ZB = 2X^2 - X^4/60^2$.

And since these two surfaces are equal,

i.e. $60X - X^2 = 2X^2 - X^4/60^2$

therefore by *al-jabr Wálmūqābalāh* we have the equation

$$3X^2 = 60X + X^4/60^2$$

or $$3X = 60 + X^3/60^2.$$

If the radius of the circle is 1 instead of 60°, then we have

$$3X = 1 + X^3$$

which is the same equation as derived by Al-Berūnī. He also computes the value of X in sexagesimal fraction as

$$X = 20^{I}\ 50^{II}\ 16^{III} 0^{IV} 45^{V}$$

*Proved earlier.

and in decimal fraction

$$X=0.3472963529. \qquad \text{(c)}$$

This value is exactly the same as claimed to be correct by Schoy.

Thus we can infer that the solution of the cubic equation $X^3+1=3X$ by Ghulām Ḥusain is more accurate than Al-Berūnī's.

Computation of Values of Sines

The computation of values of sines is based on the rule which was first given by the writers of *Siddhāntas* as the correspondence between half of a chord (*ardh jyā*) and half the angle subtended at the centre by the whole chord.[12] This rule was also adopted by other mathematicians like Al-Berūnī, Naṣiruddin Ṭūsī and Ghyāsūddīn Jamshéd. The author of the *Jāme' Bahādur-Khanī* also followed the same rule. He says that to find the value of sine of any arc, make it double and measure the length of its chord and divide it equally, one part of it will be the sine of that arc.[13] In modern terms if AB be the arc which subtended an angle ϕ at the centre of the circle of radius R and AD be the half chord (Fig. 7), then

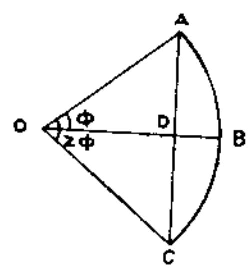

Fig. 7

$$\sin AB=AD=R\sin\phi$$

or $$R\sin\phi=\tfrac{1}{2}(AC)=\tfrac{1}{2}(\mathrm{Ch}(2\phi))=\tfrac{1}{2}(X).$$

In this connection it may be noted that Āryabhaṭa I (499 A.D.) and the author of *Pañcasiddhāntikā* divided a quadrant into 24 equal parts so that the smallest arc or angle equals 225′ and computed half chord table with the interval of 225′.[14] But there is no room for those angles which are not multiples of this interval. The reason of this is that those angles cannot be taken without trisecting an angle or arc. Although Bhāskara I (600 A.D.), Brahmagupta (628 A.D.), Bhāskara II (1150 A.D.), Nārāyaṇa (1356 A.D.) and others have given a formula in different forms for the calculation of half chord directly for the given arc or the angle,[15] in modern terms

$$\sin\phi=\frac{4\phi(180-\phi)}{40500-\phi(180-\phi)}$$

when ϕ is in degrees. But this formula gives very close approximate values. If we put $\phi=10°$, then

$$\sin 10°=0.1752577319$$

whereas the actual value of sin 10° is 0.173648177 and the above value varies after two places of decimal. Because of this rough approximation later mathematicians of India tried to trisect an arc and computed the half chord of it, by utilising this technique. Ghulām Ḥusain in his compendium *Jāme Bahādur Khanī* incorporated an exhaustive sine table. For the verification of accuracy, consider the chord length of 20° angle which was computed by the author above after deducing the cubic equation $X^3+1=3X$ from trisecting an arc which subtend an angle of 60° on the centre of the circle of radius 60°.

If radius is 1, then $X=20^{I}\ 50^{II}\ 16^{III}0^{IV}45^{V}$

$=0.3472963529.$

Therefore $\sin 10^\circ=\frac{1}{2}(X)$

$=\frac{1}{2}(0.3472963529)$

$=0.1736481764,$

which is correct up to eight places of decimal and is lesser than the actual value by the fraction of 6×10^{-10} but this lesser quantity is approximately six part out of one thousand million. So the difference is not perceptible.

Conclusion

We can say in conclusion that the construction given by Ghulām Ḥusain for trisecting an arc or angle is theoretically sound and practicable. The value of sine is also correct up to many sexagesimal places.

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