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A RATIONALE OF BHAṬṬA GOVINDA'S METHOD FOR SOLVING THE EQUATION $ax-c=by$ AND A COMPARATIVE STUDY OF THE DETERMINATION OF 'MATI' AS GIVEN BY BHĀSKARA I AND BHAṬṬA GOVINDA

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Indian scholar Bhaṭṭa Govinda (*c.* 850) perhaps used the method of continued fraction to find out the integral solution of the indeterminate equation of the type $ax-c=by$, i.e. Bhaṭṭa Govinda perhaps used the tool $p_nq_{n-1}-q_np_{n-1} =(-1)^n$ of the continued fraction for the solution of the indeterminate equation of the type $ax-c=by$. The paper presents the original Sanskrit verses (in Roman character) from Bhaṭṭa Govinda's Work, its English translation with modern interpretation. Lastly, we have considered a comparative study of the determination of *Mati* as given by Bhāskara and Bhaṭṭa Govinda.

INTRODUCTION

Bag[1] and Majumdar[2] discussed the method of integral solution of indeterminate equation of the type $by =ax\pm c$ in ancient and medieval India. But they never went through the detailed study of Bhaṭṭa Govinda's method for obtaining general solution of linear indeterminate equation of the type $by=ax-c$. In this paper it is shown that the method given by Bhaṭṭa Govinda is slightly different from that of Bhāskara I though Bhaṭṭa Govinda took the same equation $ax-c=by$ where $a<b$, i.e. after showing the result $p_nq_{n-1}-q_np_{n-1}=(-1)^n$ of the continued fraction was implicitly involved in the method of solution of the equation $by=ax-c$ and then a comparative study of the determination of 'Mati' as given by Bhāskara I and Bhaṭṭa Govinda.

2. RULE

bhājyaṁ nidhāya tadadho hāraṁ ca punaḥ parasparaṁ chindyāt
labdhamadho'dhaḥ prathamāvaptasyādhastato'pyanyat (5)

vibhajedevaṁ yāvad bhājakabhājyavaśūnyarūpau staḥ
matikalpanā ca vidhinā same pade vyatyayādviṣame (6)

bhājyādbhājyāhṛtagataśeṣonād bhājakābhihatadehāt
gatasahitād bhājyāptaṁ gatasya hānau matirbhavati (7)

ruponahāraguṇitādgantavyāptasya bhājyalabdhasya
hārahṛtasya ca śeṣsaṁ yoge hāro materaśeṣe (8)

matihatabhājyācchodhyaṁ gatamagataṁ yojayettato vibhajet
hāreṇa matiṁ vallyā'dho'dho nidhāyāptamapyasyaām (9)

upariṣṭhamupāntyahataṁ yutamantyenaivamava pare taśca
evaṁ tāvat kuryādyāvad dvāveva tau rāśī (10)

uparistho hartavyo hāreṇādhaḥsthitaśca bhājyena
s'eṣaṁ dinādi cakrādi cakrādi ca tat syādyacca tenāptam (11)

Shukla[3] translates these verses as follows:

Set down the dividend and underneath that (dividend set down) the divisor and then perform their mutual division. Write down the quotients (of mutual division) one below the other, the second one under the first, the third one under the second and so on. Carry on the mutual division till the (reduced) dividend and the (reduced) divisor are different from zero. If the number of quotients (thus obtained) is even, obtain the (number called) 'mati' in accordance with the (following) rule, and if the number of quotients is odd, obtain the 'mati' contrarily.

When the interpolator is negative, divide the interpolator by the (reduced) dividend, then subtract the resulting remainder from the (reduced) dividend, then multiply the remainder obtained by the (reduced) divisor, then increase the resulting product by the interpolator, and then divide the resulting sum by the (reduced) dividend the quotient (obtained) is the 'mati'.

When the interpolator is positive, diminish the (reduced) divisor by one (*ruponahāra*), by that multiply the interpolator, divide that by the (reduced) dividend then divide the (resulting) quotient by the (reduced) divisor, the remainder (obtained) is the 'mati'. In case the remainder is zero, then the divisor itself is the 'mati'.

Multiply the (reduced) dividend by the 'mati', then subtract the *gata* (i.e. negative interpolator) from or add the *gantavya* (i.e. positive interpolator) to that (product) and then divide the (difference or sum) by the (reduced) divisor. Write down the 'mati' under the chain (of quotients) and underneath that ('mati') write down the quotient (obtained) also.

By the penultimate number (of the chain of quotients) multiply the upper number and (to the product) add the last (i.e. lower most) number. (After doing this rub out the last number). Repeat this process again and again until there are left only two numbers in the chain.

(Of these two numbers) divide the upper number by the divisor and the lower number by the dividend (if it is possible). The remainders (obtained) denote (respectively) the days, etc. and the revolutions etc. which are the requisite quantities.

3. Rationale of the Rule

The equation is of the type $ax-c=by$ ....(1) where a=dividend, b=divisor, and remember $a<b$. Now according to the meaning of the verses, we have

$a/b)\ a\ (O=a_1$

$\quad\underline{\cdots}$

$\quad a)\ b\ (a_2$

$\qquad\underline{\cdots}$

$\qquad r_1)\ a\ (a_3$

$\qquad\quad\underline{\cdots}$

$\qquad\quad r_2)\ r_1\ (a_4$

$\qquad\qquad\underline{\cdots}$

$\qquad\qquad r_3)\ r_2\ (a_5$

$\qquad\qquad\quad\underline{\cdots}$

$\qquad\qquad\quad r_4$

Here
$$a = a_1b+a$$
$$b = a_2a+r_1$$
$$a = a_3r_1+r_2$$
$$r_1 = a_4r_2+r_3$$
$$r_2 = a_5r_3+r_4$$

Consider even number of (partial) quotients say 4 [Remember that Datta and Singh[5] said ..... So the first quotient of mutual division of a by b is always zero. This has not been taken into consideration, therefore a_5 is the even (partial quotient)]. Now according to the translation, we have,

$c/r_4)\ c\ (s$

$\quad\underline{\cdots}$

$\quad t$

Now in order to obtain 'mati', we have

$$\frac{(r_4-t)r_3}{r_4} = k \ (= \text{mati}) \qquad \text{..} \quad (2)$$

and,

$$\frac{r_4k-c}{r_3} = p$$

Consider the *valli*

$$\begin{array}{r|lll} O = a_1 & a_2L+s_3 & = & U(=y) \\ a_2 & a_2s_3+s_2 & = & \mathrm{L(x)} \\ a_3 & a_3s_2+s_1 & = & s_3 \\ a_4 & a_4s_1+k & = & s_2 \\ a_5 & a_5k+p & = & s_1 \\ k & & & \\ p & & & \end{array}$$

Here,

$$s_1 = a_5k+p = a_5k+(r_4k-c)/r_3$$

$$= \frac{k(a_5r_3+r_4)-c}{r_3}$$

$$= \frac{kr_2-c}{r_3}$$

$$s_2 = a_4s_1+k = a_4\left[\frac{kr_2-c}{r_3}\right]+k$$

$$= \frac{k(a_4r_2+r_3)-a_4c}{r_3}$$

$$= \frac{kr_1-a_4c}{r_3}$$

$$s_3 = a_3s_2+s_1 = a_3(kr_1-a_4c)/r_3+(kr_2-c)/r_3$$

$$= \frac{k(a_3r_1+r_2)-c(a_3a_4+1)}{r_3}$$

$$= \frac{ka-c(a_3a_4+1)}{r_3}$$

$$\mathrm{L} = a_2s_3+s_2 = a_2\left[\frac{ka-c(a_3a_4+1)}{r_3}\right]+\frac{kr_1-a_4c}{r_3}$$

$$= \frac{k(a_2a+r_1)-c(a_2a_3a_4+a_2+a_4)}{r_3}$$

$$= \frac{kb-cq_4}{r_3}$$

$$U = a_1 L + s_3 = a_1 \left[\frac{kb - c(a_2a_3a_4 + a_2 + {}_4a)}{r_3} \right] + \frac{ka - c(a_3 a_4 + 1)}{r_3}$$

$$= \frac{k(a_1b + a) - c(a_1a_2a_3a_4 + a_1a_4 + a_3a_4 + a_1a_2 + 1)}{r_3}$$

$$= \frac{ka - cp_4}{r_3}$$

$U/L = (ka - cp_4)/(kb - cq_4)$

and, $p_5/q_5 = a/b$, $\therefore p_5 = a, q_5 = b$.

$$\begin{aligned} \text{Now } p_5L - q_5U &= a(kb - cq_4) - b(ka - cp_4) \\ &= abk - caq_4 - abk + bcp_4 \\ &= -c(p_5q_4 - q_5p_4) \\ &= -c(-1)^5 \quad [\text{Since } p_5q_5 - q_5p_4 = (-1)^5] \\ &= c. \end{aligned}$$

We have taken $L = x$, $U = y$.

$$\begin{aligned} \text{Now we have, } p_5L - q_5U &= c. \\ \text{or, } \quad p_5x - q_5y &= c \\ \text{or, } \quad ax - by &= c \\ \text{or, } \quad ax - c &= by \end{aligned}$$

Now if we consider odd number of (partial) quotient we arrive at the same result.

Thus from the above discussion, we see that the result $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$ of the continued fraction is implicitly involved in the Bhaṭṭa Govinda's method of solution of indeterminate equation of the first degree of Bhaṭṭa Govinda used the method of continued fraction in order to solve indeterminate equation of the first degree and have an idea with the result $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$.

4. Dissimilarities

Bhāskara I and Bhaṭṭa Govinda discussed the method of solution of the same equation, viz. $by = ax - c$ where $a < b$ and their method of solution is nearly the same. The only difference is in the determination of 'mati' for the method of solution of the equation $by = ax - c$.

Now let us determine 'mati' as found by Bhāskara I. Bhāskara I said[5] "..... Now find by what number the last remainder should be multiplied, such that the product being subtracted by the (given) residue of the revolution will be exactly divisible (by the divisor corresponding to that remainder) ....." Majumdar determined mati as follows (according to the translation).

Equation being $ax-c=by$ where $a<b$.

$a/b)\ a\ (O=a_1$

$\overline{\ldots}$

$a)\ b\ (a_2$

$\overline{\ldots}$

$r_1)\ \overline{a}\ (a_3$

$\overline{\ldots}$

$r_2)\ r_1\ (a_4$

$\overline{\ldots}$

$r_3)\ r_2\ (a_5$

$\overline{\ldots}$

r_4

Consider even number of (partial) quotient, say 4. (Remember that Datta and Singh said ...... So the first quotient of mutual division of a by b is always zero. This has not been taken into consideration.) Therefore a_5 is the 4th quotient.

Let t_1=optional number (='mati'),
then according to Bhāskara I we have.

$$\frac{r_4 t_1 - c}{r_3} = k$$

or, $t_1=(k_1 r_3+c)/r_4$ .... (4)

Bhaṭṭa Govinda determined '*mati*' as given in (2). Thus we see that there is a difference between the two determinations of 'mati' (confirm 4 and 1) as given by Bhāskara I and Bhaṭṭa Govinda.

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[3]Shukla, K. S. *Laghu Bhāskarīya.* Lucknow University pp. 105-106.

[4]Ibid.

[5]Datta and Singh —*History of Hindu Mathematics.* (A source book) Part II (Algebra p. 99).

[6]Ibid.