ON THE SQUARE ROOT FORMULA IN THE *BAKHSHĀLĪ MANUSCRIPT*

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The *Bakhshālī Manuscript* is famous for the *Sūtra* (which we will refer in this paper as *Bakhshālī Sūtra*) for the approximate computation of square roots of non-square numbers. The purpose of this paper is four fold namely to give the correct deciphering of the *Bakhshālī Sūtra* to point out the incorrectness of the available translation of the *Sūtra* to give the correct translation of the *Sūtra* and to offer a plausible method that would have been employed in the original derivation of the rule described by the *Sūtra*.

1. Introduction

The *Bakhshālī Manuscript*(*BM*) which was unearthed by a farmer in 1881 A.D. at Bakhshālī, a village near Peshawar, contains Mathematics written on about 70 birch barks. The period of composition of the Bakhshālī Mathematics is placed in the range 200-400 A.D., the details about which can be found in B. Dutta[1] and Martin Levey and Marvin Petruck[2]. The Manuscript was edited, with an elaborate introduction, by G. R. Kaye[3] and published in 1927. The highly biased and hostile views of Kaye expressed in his introduction have been extensively examined and rebutted by B. Dutta[4] in 1929.

An item of great historical importance in the *BM* is the formula for computing the square roots of non-square numbers, namely

$$\sqrt{Q} = \sqrt{A^2+b} = A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)}, \qquad .. \quad (1.1)$$

which we will call, hereafter, the Bakhshālī formula.

The formula (1.1) is certainly a refinement over

$$\sqrt{Q} = \sqrt{A^2+b} = A + \frac{b}{2A} \qquad .. \quad (1.2)$$

which is known as Heron's formula, after the Greek mathematician Heron who lived in the second half of the first century A.D.[5]. The formula (1.2) was evidently known to the Bakhshālī Author (BA). The only point in support of the argument that the formula (1.2) might have reached India from Greece is that there was trade connection between the two countries from very early times. But the following points serve to support the argument that the formula (1.2) was an independent Indian discovery :

(*i*) Techniques for approximate computation of square roots were known to Indians centuries before Christ. For instance, the Jainas of India who approximated (of course wrongly) π by $\sqrt{10}$ were able to obtain $\sqrt{10}$ correct to 13 decimal places[6].

(*ii*) In Heron's work, the formula (1.2) occurs in connection with a geometrical problem[7], while in the Bakhshālī Mathematics, the square root formula occurs in connection with an algebraic problem.[8] This is highly characteristic of the development of early Greek and Indian Mathematical thoughts, which developed independently, the Greeks specialising in Geometry and the Indians in Algebra.

The aim of the present investigation is to give the correct deciphering of the *Bakhshālī Sūtra* (*BS*) describing the square root formula, to critically analyse the defective translations of the *Sūtra* by Kaye and the subsequent researchers, to give the correct translation of the *Sūtra*, and to present a plausible method that would have been employed in the original derivation of the formula described by the *BS*.

2. Correct Deciphering of the *Bakhshālī Sūtra*

The birch barks on which the *Sūtra* is written are much mutilated ; but as the *Sūtra* is given on three separate occasions, the fragments can be pieced together to get the complete *Sūtra*. We first reproduce the *Sūtra* as deciphered by Kaye.[9]:

akṛte ślişṭa kṛthyūnā śeṣa cchedo dvisangunah

tadvarga dala saṃśliṣtha hṛti śuddhi kṛti kṣayah

This version of Kaye is accepted by the subsequent workers like Datta[10], Srinivasiengar[11] and others. The above deciphering needs correction to the extent that "*Kṛtyūnā s'eṣachedā*" is to be replaced by "*Kṛtyūnāt śeṣacchedo*". This is justified from the following considerations. The above portion of the *Sūtra* is fully legible in two places in the text. It occurs in one place[12] in the form "कृत्यूना शेषच्छेदो". while in the other[13] it occurs as "कृत्यूनात् शेषच्छेदो" We can even explain the discrepancy by accepting the contention of B. Datta[14] that the available *BM* is not the original composition but only a copy. The 'scribe' (not the author) who made the copy might have omitted the letter 'त्' in the first case. Such a thing can certainly happen particularly in a situation where one recites and the other scribes. We accept the second one as correct as it makes the *Sūtra* meaningful. The first version "कृत्यूना शेषच्छेदो" represents an impossible construction because "कृत्यूना" being in feminine form cannot be taken as an adjective of "शेषच्छेदः" which is masculine; nor can it be construed as a noun in the feminine—nominative—singular form, because it then becomes impossible to interpret it meaningfully. We thus obtain the following as the correct version of the *Sūtra* :

अकृते श्लिष्टकृत्युनात् शेषच्छेदो द्विसंगुणः ।
तद्वर्गदल संश्लिष्टहृति शुद्धिकृति क्षयः ॥

3. Kaye's Translation of the *Sūtra*

Kaye[15] translates his version of the *Sūtra* as :

"The mixed surd is lessened by the square portion and the difference divided by twice that. The difference is divided by the quantity and half that squared is the loss", which according to Kaye, corresponds to (1.2). As B. Datta[16] rightly points out, the above translation of Kaye is "wrong and meaningless", because the correct translation of the *Sūtra* should correspond to (1.1) as we shall see later. Here, we shall only point out the inconsistencies that Kaye displays an account of his wrong translation. It is very difficult to understand how Kaye failed to doubt the correctness of his translation in view of the following facts :

(*a*) The *BM* contains many numerical examples involving the application of the formula (1.1). For instance, in the evaluation of $\sqrt{41}$ the following detailed "working" is shown :

$$\sqrt{41} = 6+\frac{5}{12}-\frac{\overline{144}^{\,25}}{2\left(6+\frac{5}{12}\right)} \qquad \ldots \quad (2.1)$$

This is preceded by the statement of *BS* for square roots. (2.1) is nothing but (1.1) with $Q = 41$, $A = 6$, and $b \equiv 5$. It is to be noted that if the *BA* had meant (1.2) by his *Sūtra,* he would not have quoted it everytime and then used (1.1) in the immediately following example.

(*b*) While making the general comments on the nature of the contents of the text, Kaye[16] speaks of the "........rather special characteristics of the work, namely the apparently over-elaborated exposition of the "workings" of the solutions to the preservation of the generality of solutions". Thus the "working" given in connection with the computation of $\sqrt{41}$ described above should have provided him with a clue to the correct form of the formula described by the *BS*.

(*c*) While describing the method of solution for quadratic indeterminate equations, Kaye[17] makes the observation "No general rule is preserved but the solution itself indicates the rule".

But Kaye fails to apply the above logic to the *BS* for square roots. He thus violates his own norms in compromising with his wrong translation of the *Sūtra.*

Having translated the *Sūtra* wrongly, and having had to explain the applied forms of the *Sūtra* like (2.1), Kaye tries to wriggle out of the situation in the following manner. We quote him[18]: "The rule means that the first approximation to $\sqrt{Q}=\sqrt{A^2+b}$ is $A+\frac{b}{2A}$ or q_1; but $q_1^2-Q = \left(\frac{b}{2A}\right)^2 = e_1$. No rule for the 'second approximations' is preserved but there are several examples; and of course, no fresh rule is really required, for

$$\sqrt{Q} = \sqrt{q_1^2-e_1} \sim q_1 - \frac{e_1}{2q_1} = A + \frac{4A^2b+b^2}{8A^3+4AB}\text{''}. \qquad .. \quad (2.2)$$

He further makes the comment:

"Note that if $\sqrt{A^2+b} = r$, then

$$r = A + \frac{b}{2A+}\;\frac{b}{2A+}\;\cdots\cdots$$

The text uses

$$r^1 = A + \frac{b}{2A}, \quad \text{and} \qquad .. \quad (2.3)$$

$$r^{111} = A + \frac{b}{2A+}\;\frac{b}{2A+}\;\frac{b}{2A} = A + \frac{4A^2b+b^2}{8A^3+4Ab} \qquad .. \quad (2.4)$$

The above argument of Kaye is self-contradicting because, soon after making the statement "No fresh rule is really required", he proceeds to actually derive the rule! He does it in two ways as indicated by (2.2) and (2.4).

In obtaining (2.2) he writes $\sqrt{q_1^2-e_1} \sim q_1 - \frac{e_1}{2q_1}$. He does not explicitly mention how he approximates $\sqrt{q_1^2-e_1}$ by $q_1-(e_1/2q_1)$. Evidently, two ways are open to him.

(*i*) He can use the first order formula (1.2). But in the *BS*, (1.2) is found used only for the case where the non-square number is expressed as the "sum" of two quantities like A^2+b. This is conceded by Kaye himself. Dealing with an example found in the Manuscript he writes[19]".

$$\text{``}\sqrt{481} = \sqrt{21^2+40}\,; \quad q_1 = 21\frac{40}{42}\text{''},$$

and makes the following comment in a foot note:

"The form $$\sqrt{481} = \sqrt{22^2-3} \sim 22 - \frac{3}{44} = 21\frac{41}{44}$$

suggests itself; but the negative sign does not appear to have been used in the first approximations".

We wish to point out that there is a numerical error in the text (edited by Kaye) where $21\frac{37}{44}$ is found printed in place of the correct value $21\frac{41}{44}$ which has been incorporated into the reproduced foot note given above.

(*ii*) He can use the Binomial theorem for fractional indices and write

$$\sqrt{q_1^2-e_1} = q_1\left(1-\frac{e_1}{q_1^2}\right)^{1/2} \sim q_1\left(1-\frac{e_1}{2q_1^2}\right) = q_1-\frac{e_1}{2q_1}.$$

We have to point out that Binomial theorem was, known to ancient Indians, but only for the positive integral index[20].

In obtaining (2.4), Kaye has used continued fractions. But there is no clear evidence that ancient Indians before Bhāskara knew anything about continued fractions[21].

Thus it is clear that the *BA* cannot have obtained (1.2) from any one of the above three methods because none of the three tools—first order formula (1.2) with negative sign, the Binomial theorem for a fractional index or continued fractions—was available to him. We will come back to this point in section 7.

4. Datta's Translation of the *Bakhshālī Sūtra*

B. Datta[22] points out the incorrectness of Kaye's translation, and rightly recognises the *Sūtra* as representing the second approximation (1.1) and not the first approximation (1.2). He gives his own translation of the *Sūtra* (as deciphered by Kaye) in the following form:

"In case of a non-square (number) subtract the nearest square number; divide the remainder by twice (the root of that number). Half the square of that (that is, the fraction just obtained) is divided by the sum of the root and the fraction *saṃśliṣṭa* and subtract; (this will be the approximate value of the root) less the square (of the last term)".

In order to identify the defects in Datta's translation, we try to identify the several portions of the formula (1.1) that follow from his translation.

According to Datta's translation,

(*i*) अकृते = In the case of non-square number;

$$A^2+b.$$

(*ii*) श्लिष्टकृत्यूना = subtract the nearest square number;

$$A^2+b-A^2 = b.$$

Here the word "कृत्यूना" with its feminine form does not allow itself to be associated with any other word in the *Sūtra*, and thus is impossible of any meaningful translation. This fact has, evidently, been ignored by Dutta.

(*iii*) शेषच्छेदोद्विसंगुणः=divide the remainder by twice (the root of that number). It is impossible to arrive at this translation, because the actual meaning of words is "The denominator or the divisor of the remainder (शेषस्यछेदः = शेषच्छेदः) is multiplied by two". This step is supposed to yield $b/2A$.

(*iv*) तद्वर्गदल संश्लिष्टहृतिः = Half the square of that is divided by the sum of the root and the fraction;

$$1/2\,(b/2A)^2/(A+b/2A)$$

(*v*) शुद्धिः = and subtract.

Here the subtrahend is clearly the quantity got in stage (*iv*). But the translation does not specify the quantity from which the subtraction is to be carried out. If it is taken as $A+\frac{b}{2A}$, we arrive at the formula (1.1).

(*vi*) "less the square (of the last term)", which evidently is the translation of "कृतित्यजः". But this fails to convey any meaning whatsoever.

The above analysis shows that Datta's translation with its correct aim of identifying the *Sūtra* with (1.1), leaves much to be explained.

5. C. N. Srinivasiengar's Translation

C. N. Srinivasiengar translates the *BS* in the following terms[23]:

"In the case of a non-square number, subtract the nearest square number; divide the remainder by twice (the nearest square); half the square of this is divided by the sum of the approximate root and the fraction. This is subtracted and will give the corrected root".

We would only remark that this translation is almost the same as Datta's except that it is free from the mistake analysed under the stage (*vi*) in discussing Datta's translation.

6. Correct translation of the *Bakhshālī Sūtra*

We give below the correct translation of the *Sūtra*. We first give the meaning of each word in the *Sūtra*, and then the complete translation.

At the outset, we wish to point out that the two words कृति and ऊन are to be interpreted in a slightly unconventional way.

The word कृति is used by the later writers like Āryabhaṭa (5th century A.D.), Bramhagupta (7th century A.D.) and Bhāskara (12th century A.D.) as a synonym to the word वर्ग (square). We have to note that the ancient Indian Mathematicians were given to write mostly in verses and it was, therefore, a matter of convenience for them to use different words to mean the same arithmetical process. For instance, संकलन, युति, यौग are used to denote addition; व्यवकलन, विशुद्धि, शोधन are used to denote subtraction and भाजन, हरण, छेदन to denote division. In each of the above three sets of synonyms, we easily see the property that each word has something inherent in it which can be associated with the process it is meant to describe. But the word कृति has nothing inherent in it to mean "the product of two numbers which are equal (वर्ग)". It is derived from the root कृ (डु कृञ् करणे) which means "to do, to perform, to work......". Thus कृति can mean "a deed or process'. Hence the meaning वर्ग to the word कृति is purely an assigned one. It is, therefore, natural that Bhāskara in his *Līlāvatī*[24] finds it necessary to give the definition ""समद्विघात: कृतिरुच्यते"" before using the word कृति to denote वर्ग.

The *BA*, coming much before Bhāskara, Brahmagupta and Āryabhaṭa, was naturally not bound by this tradition of using the word कृति to mean वर्ग. It is our assertion that he has used the word कृति as a synonym to the word "वर्गमूल" which means the "square root". This assertion is not merely conjectural, but is based on the following evidences from the text:

(*i*) The *BA* uses the word कृति in the *BS* and no where else.

(*ii*) He has occasion to deal with the process of "squaring" at least half a dozen times when he discusses वर्गगणित[25]. But in all those places he uses invariably the word वर्ग. The word कृति is not used even once to denote a square.

(*iii*) The *BS* does not contain the word मूल. If the word कृति is not taken to mean मूल, the *Sūtra* would be incomplete in the sense that it will be failing to mention its own object namely of being meant for computing the square root.

(*iv*) In "*Śulba Sūtra*" whose period of composition is placed in the range 800—500 B.C., the word करणि is used to denote the "square root" of a given number[26]. For example, we find the following usages: द्विकरणि $= \sqrt{2}$, त्रिकरणि $= \sqrt{3}$, तृतीयकरणि $= 1/\sqrt{3}$ etc. It is significant that the word करणि is derived from the root कृ which is also the root for the word कृति. In fact, the dictionary compiled by C. Srinivasa Gopalacharya[27] lists, among other things, the word कृति as a synonym to the word करण. It is thus very clear that the *BA* has used कृति (the synonym of करणि) to denote the square root of a number. He has preferred the word कृति to करणि, perhaps, for metrical reasons.

Before giving some more arguments in support of the above assertion, it is necessary to consider the word ऊनात्.

The word ऊन is normally used to denote the operation of subtraction. But in the *BS* it is used, we assert, to denote the operation of division. This assertion is made on the following grounds:

(*i*) The word शेष is used by the *BA* in connection with two arithmetical processes: (*a*) in subtraction to denote the result of subtracting one quantity from another, (*b*) in division to denote the ratio of the remainder to the divisor. We quote an instance of the usage (*b*) from the text. In dealing with a problem on "computation of gold"[28] the *BA* deals with the division of 330 by 45 which he describes in the following manner:

| 330 / 45 | *Pañcahdaśabhāgeccheda* | *Phalam* | 7 śe 1 / 3 |.

Thus, the word शेष refers not to the usual remainder (15) but to the ratio of the remainder (15) to the divisor (45). This special sense in which the word शेष is used in the *BM* is very crucial to our discussion and we will refer to this aspect later.

Since the *sūtra* contains the phrase शेषच्छेद: which can only be understood as शेषस्यच्छेद : (the denominator of the *śeṣa*), and since this शेष is obtained as a result of the operation described by " ऊनात् ", we have to take the word ऊन to mean division, so that the form ऊनात् can be taken to mean "by division".

(*ii*) The word हरण which is used to denote the operation of division, and the word परिहा which means "to diminish, decrease, decay" are both derived from the root " ह ". The word ऊन is used to mean " परिहा " as indicated by the Pāṇini *sūtra* " ऊन " (परिहाणे), घस् " (३ — ३ — २६), which also shows that the word is to be used in masculine. This, incidentally, supports the form " ऊनात् " found in our deciphering of the *BS* as against the form " ऊना " found in Kaye's transliteration.

(*iii*) With the words कृति and ऊन taking the meanings as explained above, "कृत्यूनात् " (कृत्या+ऊनात् तृतीय तत्पुरुष) would mean "by division by the (approximate) root". We, thus, get the result of the first step in the computation of the square root of A^2+b as $\frac{A^2+b}{A} = A+\frac{b}{A}$, where $\frac{b}{A}$ (and not b) is described by the word शेष . The next step is described by " शेषच्छेदो द्विसंगुण:" which means "the denominator of the *śeṣa* is multiplied by two", which gives the result of the second step in the form $A+\frac{b}{2A}$. That these are the first two steps contemplated by the *BS* is confirmed by the "workings" of numerical examples contained in the text. We quote two examples :

(*a*) While computing $\sqrt{41}$, the *BA* records[29] both the following steps :

$$\left|\begin{array}{c} 6 \\ 5 \\ 6 \\ \hline \end{array}\right| \text{ and } \left|\begin{array}{c} 6 \\ 5 \\ 12 \\ \hline \end{array}\right| \text{which mean } 6+\frac{5}{6} \text{ and } 6+\frac{5}{12}.$$

(*b*) Again while dealing with the computation of 481, both the above steps are given. But in this case there is some difficulty in deciphering the material in the original. The leaf (the birch bark) on which the material is written is much mutilated and the different pieces into which it has broken, have been pieced together. This version is found in the facsimile of the leaf given in Kaye[30]. According to Kaye's transliteration[31] of the material contained in the above leaf, the results of the first two steps in the computation of $\sqrt{481}$ are recorded as $\left|\begin{array}{c} \mathit{20} \\ 21 \\ \hline \end{array}\right|$ and $\left|\begin{array}{c} 21 \\ 20 \\ 21 \\ \hline \end{array}\right|$, where the dotted parts of the boundary of the first block indicate that there is joining of two broken pieces, and the italics used in writing 20 in the first block indicate "doubtful reading". We wish to point out that the correct reading of the figures in the first block should be $\begin{array}{|c|}40\\21\\\hline\end{array}$, because according to the *BS* (as explained above) the result of the first step in the computation of $\sqrt{481}$ should be $\begin{array}{|c|}21\\40\\21\\\hline\end{array}$. The top-most portion of leaf (where the above step is recorded) being not legible, we are able to get only the reading $\begin{array}{|c|}40\\21\\\hline\end{array}$. We further wish to point out that Kaye would have got, even for a doubtful reading, $\begin{array}{|c|}40\\21\\\hline\end{array}$ if he had recognized the first step in the computation of $\sqrt{A^2+b}$ to be $(A^2+b)/A$.

We see from the above that we can explain the steps $A+\frac{b}{A}$ and $A+\frac{b}{2A}$ in the worked examples of the text by taking the words कृति and ऊन to denote the "square root" and "division" respectively, which is not possible if we take them to denote "square" and "subtraction".

(*iv*) Since there is the description of the " छेद " of " शेष ", the result शेष can only pertain to the result of the operation of division. If we were to interpret शेष to mean the result of subtraction (as done by earlier workers), we would have the शेष : $A^2+b-A^2=b$. To speak of the छेद of this quantity would be irrelevant. Even if one argued that it has the छेद 1, then the result of the second step in the computation of $\sqrt{A^2+b}$ would be $b/2\times 1$ and not $b/2A$. Thus the word ऊनात् has to be taken to mean division.

We would finally point out that with the new meanings assigned to कृति and ऊन , we can satisfactorily translate every word in the *BS*.

We next consider the word अकृते. It is the locative singular form of the noun अकृत which is in neuter gender, and which means "the lack of work". The work involved here is the extraction of square root. Thus अकृते is to be understood in the sense : "(मूलं) अकृते "which can be translated into "In the case of a number whose square root is not found" or in other words "In the case of a number whose square root is to be found".

The word शुद्धि is normally used to denote the operation of subtraction. But in the *BS*, we have to take it to stand for its natural meaning "refinement or correction". The fourth quarter of the *BS* contains the words " शुद्धिकृति क्षयः ". Here we have to take the combination शुद्धिकृतिः to mean the refined root as against the श्लिष्टकृतिः which stands for the approximate root. The word क्षयः can be taken to have its usual meaning, namely subtraction. Datta gets into trouble by taking the combination " कृतिक्षयः " and translates it into "less the square (of the last term)" which fails to convey any meaning. His difficulty is inevitable if कृति is taken to mean square.

The word संश्लिष्ट is used to denote the first approximation to the square root given by (1.2). This is clear from the following. In connection with the computation of $\sqrt{889}$, the *BA* states the *BS* and addresses the exercise to the student in the terms[32] " अनेन सूत्रेण श्लिष्टमूलमानयस्व मतिमा (न्)"* and immediately gives the answer (without giving the "working" in the form $\begin{array}{|c|} 29 \\ 48 \\ 58 \\ \hline \end{array}$ which is just the first approximation (1.2) for $\sqrt{889}$. Thus श्लिष्ट or संश्लिष्ट is used to denote the first approximation to the root. Here, we wish to point out that the above way of addressing the student " अनेन ...." is not correctly understood by Kaye. He, therefore, translates the above, wrongly into[33] "by means of this rule an approximation (*Ānaya*) to the proper root of mixed quantity is found". We now put down together the meanings of different words of the *BS* as follows :

अकृते = In the case of a number whose square root is to be found, श्लिष्टकृति: = approximate root, ऊनात् = by division, शेष: = the ratio of the remainder to the divisor (In the translation also, we shall agree to call this by the name *śeṣa* for want of the English equivalent) ेद: = denominator, द्विसंगुण: = multiplied by two, तद्वर्ग = square of that, दल = half, संश्लिष्ट = composite fraction (which is the first approximation to the root), हृति: = division, क्षय: = subtraction, शुद्धिकृति: = refined root.

Translation of the Sūtra

"In the case of a number whose square root is to be found, divide it by the approximate root (the root of the nearest square number); multiply the denominator of the resulting *śeṣa* (the ratio of the remainder to the divisor) by two; square it (the fraction just obtained); halve it; divide it by the composite fraction (the first approximation); subtract (from the composite fraction); (the result is) the refined root".

The above algorithm applied to the computation of $\sqrt{A^2+b}$ yields the following sequence of steps :

$$\frac{A^2+b}{A},\ A+\frac{b}{A},\ A+\frac{b}{2A},\ \left(\frac{b}{2A}\right)^2,\ \frac{1}{2}\left(\frac{b}{2A}\right)^2,$$

$$\frac{\left(\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)},\ A+\frac{b}{2A}-\frac{\left(\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)}$$

***The way the teacher addresses his student is remarkable. It shows that at the advanced level of learning the teacher-pupil relationship was based on affection and mutual respect.**

7. A Plausible Method of the Derivation of the Bakhshālī Formula

The formula (1.1) is quoted by many writers of History of Mathematics like B. Dutta[34], Martin Levy and Mervin Petruck[35], D. M. Bose *et al.*[36] D. E. Smith[37] C. N. Srinivasiengar[38],[39] But to the author's knowledge, no where it is mentioned as to how the formula (1.1) might have been originally derived. In fact, C. N. Srinivasiengar[40] observes "........we have yet to obtain a convincing explanation as to how the *BA* got the square-root formula (1.1)........".

We give below, a plausible method which would have been employed by the *BA* to obtain (1.1). The method is based on the principle of iteration. It is certainly difficult to say as to from which time the simple idea of iteration came to be used in Mathematics and in particular in Numerical analysis. All that we can say is that it must be a very old idea for it can be seen to represent one of the basic human instincts namely of improving a given thing based on one's previous experience. Moreover, the application of this simple idea does not call for the knowledge of any other sophisticated mathematical equipment. So, it is reasonable to think that the *BA* was familiar with the process of iteration. Thus, he could have arrived at (1.1) in the following way.

Let the approximation to $\sqrt{Q} = \sqrt{A^2+b}$ as given by (1.2) be denoted by q_1, so that

$$q_1 = A + \frac{b}{2A} \quad .. \quad (3.1)$$

A refined value q_2 of the root can be obtained by replacing the initial approximation A in (3.1) by q_1. Thus

$$q_2 = q_1 + \frac{b_1}{2q_1}\ , \quad .. \quad (3.2)$$

where

$$Q = q_1^2 + b_1\ . \quad .. \quad (3.3)$$

Substituting for b_1 from (3.3) we get (3.4)

$$q_2 = q_1 + \frac{Q-q_1^2}{2q_1} \quad .. \quad (3.4)$$

Now substituting for q_1 from (3.1) and replacing Q by

A^2+b, we get

$$q_2 = A + \frac{b}{2A} + \frac{A^2+b-\left(A+\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)}$$

simplifying which we get (1.1).

We wish to point out that the above procedure of obtaining (1.1) is different from that used by Kaye in obtaining (2.2) in the sense that we use here the first order formula (1.2) involving the "positive sign", while Kaye uses the same formula with the "negative sign", which as shown in our Analysis (section 2), was not known to the Bakhshālī author. We have thus shown that Bakhshāli formula comes out in a very simple and natural manner when we apply the process of iteration to the first order formula (1.1) involving only the "positive sign". In the absence of other methods of derivation consistent with the mathematical knowledge of those times, the method of iteration as explained above seems to be the only method used to obtain (1.1). We may, in a lighter vein, recall here the famous quotation of Sherlock Holmes: "........when you have eliminated all which is impossible, then whatever remains however improbable, must be the truth".

8. Conclusion

Investigations in this paper have succeeded in (*i*) correctly deciphering the *Bakhshālī Sūtra* for square roots, (*ii*) highlighting the incorrectness of the available translations of the *Sūtra* (*iii*) providing the correct translation of the *Sūtra*, (*iv*) offering a plausible method for the original derivation of the Bakhshālī square root formula.

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