USE OF HYPOTENUSE IN THE COMPUTATION OF THE EQUATION OF THE CENTRE UNDER THE EPICYCLIC THEORY IN THE SCHOOL OF ĀRYABHAṬA I ???

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The present paper refutes the assertion of T. S. Kuppanna Shastri that the use of the hypotenuse in the computation of the equation of the centre under the epicyclic theory is one of the principal characteristics of the school of Āryabhaṭa I. It has been shown that the followers of Āryabhaṭa I, like other Hindu astronomers, did not employ the hypotenuse in calculating the equation of the centre under the epicyclic theory. The reason for not using the hypotenuse is explained and the views of the prominent Hindu astronomers, such as Bhāskara I, Govinda Svāmi, Parameśvara, Nīlakaṇṭha, and others are cited in support.

1. INTRODUCTION

T. S. Kuppanna Shastri in a paper entitled "The school of Āryabhaṭa and the peculiarities thereof" published in an earlier issue of this Journal[1] has proclaimed that the use of the hypotenuse in the computation of the equation of the centre under the epicyclic theory is an important characteristic of the school of Āryabhaṭa I. Writes he :

> "Another important peculiarity of this school is the use of the true hypotenuse in the computation of the equation of the centre. The use of the hypotenuse in the equation of conjunction is common and accepted by all schools, as justified by the eccentric or epicyclic theory of the motion of the planets, which can be readily seen from a geometrical representation of the motion. By the same logic, the hypotenuse should be used for the equation of the centre also, the theory being essentially the same. That is why this school uses it, as a geometrical consequence of this theory set forth by Āryabhaṭa in *Kālakriyā :* 17-21, combined with the theory of uniform motion given in *Kāla :* 12-14. Thus, in the ***Mahābhāsk.***, IV, 8-12, the manner of getting the true hypotenuse as based on the theory of epicycles is given, and in 19-20 the same as based on the eccentric theory. In 21, the approximate sine equation of the centre is asked to be multiplied by the radius and divided by the true hypotenuse to get the correct sine equation of the centre. *Vaṭ. Sid. Spaṣṭādhikāra*, II, 3-4, gives the method of getting the true hypotenuse, and III, 11 instructs its use to divide the approximate equation of the centre to get the correct one.

The use of the hypotenuse is not only a logical result of the theory, but it will also give a better result. It supplies part of the second term of the modern correct equation of the centre. Neglecting powers of e (eccentricity) higher than the square, the first two terms are $2e \sin m - 5/4\, e^2 \sin 2m$, where m is the mean anomaly reckoned from the higher apsis, as in Hindu astronomy. The distance between the centres of the original and eccentric circles is equal to $2e$. It is also the radius of the epicycle. According to the theory, correct sine equation of the centre $= 2e \sin m \div h$ (=hypotenuse). But $h = \sin m / \sin (m - \text{eq. cent.})$, if the radius of the eccentric circle is taken as unity. Therefore $\sin (\text{eq. cent.}) = 2e \sin m \times \sin (m - \text{eq. cent.}) / \sin m = 2e \sin (m - \text{eq. cent}) = 2e \sin (m - 2e \sin m)$ (since the eq. cent. is small) $= 2e \sin m - 4e^2 \sin m \cos m = 2e \sin m - 2e^2 \sin 2m$. Though we get $2e^2$ as the coefficient of the second term, instead of the correct $5/4\, e^2$, it will not make much difference, being the second power of e. Also, the point is that we get the term instead of neglecting it. Using the Moon's epicycle of $31\frac{1}{2}$ degrees, which gives $7/80$ as the value of $2e$, we get for the second term—$13' \sin 2m$, the same as the modern correct one. (The apparent complete agreement is due to the Hindu coefficient of the first term being defective by about a fifth.).

Bhāskarācārya II discusses the point, why other schools do not use the hypotenuse for the equation of centre. He says that some do not use it thinking that the difference is small. This depends upon what we consider small and negligible and may be accepted. But the other argument he gives, quoting his master Brahmagupta, that the theory itself is that the epicycle, instead of being uniform, is proportionate to the true hypotenuse and has to be multiplied by it and divided by the radius, and therefore, the division by the true hypotenuse is cancelled out, is untenable, for this kind of argument helps only to shut out a tolerably good theory already existing and nothing more, and is just a way of escape, as pointed out by Caturvedācārya in his commentary on the *Brāhmasphuṭa Siddhānta* (cf. *Sid. Śiromaṇi : Gola : Chedyaka ;* and commentary thereon)."

The above statement does not reflect a correct understanding of the school of Āryabhaṭa I. In paragraph 1, Kuppanna Shastri tells us that in *Mahābhāskarīya*, iv. 21, the approximate sine equation of the centre is asked to be multiplied by the radius and divided by the true hypotenuse to get the correct sine equation of the centre. The same are stated to be the contents of *Vaṭeśvarasiddhānta*, II, iii. 11. But, contrary to what Kuppanna Shastri has said, both *Mahābhāskarīya*, iv. 21 and *Vaṭeśvarasiddhānta*, II, iii. 11 state the following formula and its application :

$$R \sin (\textit{spaṣṭabhuja}) = \frac{R \sin m \times R}{H},$$

where m is the *madhyamabhuja* (i.e. mean anomaly reduced to bhuja).

The formula

$$\sin\text{ (equation of centre)} = 2e \sin m \div h,$$

on which Kuppanna Shastri bases his conclusions in paragraph 2 does not occur even in any nook or corner of the school of Āryabhaṭa I. There is not even a smell of it. The formula which has been actually used by the followers of Āryabhaṭa I is

$$R \sin\text{ (equation of centre)} = \frac{\text{tabulated } manda \text{ epicycle} \times R \sin m}{80}, \tag{1}$$

the denominator being 80 instead of 360 because the tabulated *manda* epicycle is abraded by $4\frac{1}{2}$, or in the notation of Kuppanna Shastri,

$$\sin\text{ (equation of centre)} = 2e \sin m.$$

Kuppanna Shastri seems to have been misled by the use of the true hypotenuse (*mandakarṇa* obtained by iteration) in the formula for the planet's *spaṣṭabhuja*, viz.

$$R \sin\ (spaṣṭabhuja) = \frac{R \sin m \times R}{H}, \tag{2}$$

where H is the true *mandakarṇa* (obtained by iteration), or, in the notation of Kuppanna Shastri,

$$\sin\ (m - \text{eq. centre}) = \sin m \div h.$$

He has missed to see that equation (1) is based on the tabulated *manda* epicycle which is false (*asphuṭa*) and on which the planet does not move, whereas equation (2) relates to the true eccentric on which the planet actually moves.

Kuppanna Shastri has also misquoted Bhāskara II to suit his purpose. In the passage under reference, Bhāskara II has said that Caturvedācārya Pṛthūdaka, who held views similar to those of Kuppanna Shastri, was not correct, and that Brahmagupta, whose views have been declared to be untenable by Kuppanna Shastri, was correct.

It would be interesting to note that whereas Kuppanna Shastri declares the use of hypotenuse in the computation of the equation of the centre to be an important peculiarity of the school of Āryabhaṭa I, the great scholiasts of Āryabhaṭa I, such as Bhāskara I, Govinda Svāmi, Parameśvara and Nīlakaṇṭha, have taken pains to demonstrate why the hypotenuse has not been used in the computation of the equation of the centre.

The object of the present paper is to explain why the hypotenuse has not been used in the computation of the equation of the centre under the epicyclic theory and also to give the views of the prominent Hindu astronomers on this point.

2. Tabulated Manda Epicycles, True or Actual Manda Epicycles, and Computation of the Equation of the Centre

The *manda* epicycles whose dimensions are stated in the Hindu works on astronomy are not the actual epicycles on which the true planet (in the case of the Sun and Moon) or the true-mean planet (in the case of the star-planets, Mars, etc.) moves. Āryabhaṭa I has given two sets of the *manda* epicycles one for the beginning of the odd quadrant and the other for the beginning of the even quadrant. If one wants to find the *manda* epicycle for any other place in the odd or even quadrant, one should apply the proportion stated in *Mahābhāskarīya*, iv. 38-39(i) or *Laghubhāskarīya*, ii. 31-32. The local *manda* epicycle thus obtained is called the true *manda* epicycle (*sphuṭa-manda-vṛtta*), but this too is false (*asphuṭa*). Writes Parameśvara (1430) in his *Siddhāntadīpikā* :[2]

स्फुटितान्यपि मन्दवृत्तान्यस्फुटानि भवन्ति, तेषां कर्णसाध्यत्वात् । अतः कर्णसाधितवृत्तसाध्या भुजाकोटिफलकर्णा इति ।

i.e., "The *manda* epicycles, though made true, are false (*asphuṭa*), because the true (actual) *manda* epicycles are obtained by the use of the (*manda*) *karṇa*. Therefore, (the true values of) the *bhujāphala*, *koṭiphala* and *karṇa* should be obtained by the use of the (*manda*) epicycles determined from the (*manda*) *karṇa*."

But how are the *manda* epicycles made true by the use of the *mandakarṇa* ? Lalla (c.748) has answered this question. Says he :[3]

सूर्येन्दुमन्दगुणकौ मृदुकर्णनिघ्नौ
त्रिज्योद्धृतौ भवत एवमिह स्फुटौ तौ ।
ताभ्यां पुनश्च भुजकोटिफले विधाय
साध्ये श्रुती मुहुरतः स्वगुणौ श्रुती च ॥

i.e., "The *manda* multipliers (= tabulated *manda* epicycles) for the Sun and Moon become true when they are multiplied by the (corresponding) *mandakarṇas* and divided by the radius. Calculating from them the *bhujāphala* and *koṭiphala* again, one should obtain the *mandakarṇas* (for the Sun and Moon as before) ; proceeding from them one should calculate the *manda* multipliers and the *mandakarṇas* again and again (until the nearest approximations for them are obtained)."

The process of iteration is prescribed because the (true) *mandakarṇa* is unknown and is itself dependent on the true *manda* epicycle. If the (true) *mandakarṇa* were known, the true *manda* epicycle could be easily determined from the formula :

$$\text{true } manda \text{ epicycle} = \frac{\text{tabulated } manda \text{ epicycle} \times \text{true } mandakarṇa}{R}. \qquad (3)$$

What is true for the *manda* epicycles of the Sun and Moon is also true for the *manda* epicycles of the planets, Mars, etc.. Bhāskara II, commenting on the above passage of the *Śiṣyadhīvṛddhida* of Lalla, observes :[4]

"तथा कुजादीनां मन्दकर्मणि उक्तवत् कर्णमुत्पादयित्वा तेन स्वमंदपरिधिं हत्वा व्यासार्धेन विभजेत्, फलं कर्णवृत्ते परिधिः। तेन पुनरुक्तवद् भुजकोटिफले कृत्वा ताभ्यां मन्दकर्णमानयेत्। एवं तावत्कर्म कर्तव्यं यावदविशेषः।"

"मन्दपरिधिस्फुटीकरणं त्रैराशिकात्—यदि व्यासार्धवृत्ते एतावान् परिधिस्तत्कर्णवृत्ते कियानिति फलं कर्णवृत्तपरिधिः, कर्णवृत्तपरिधेरसकृत्करणं च कर्णस्यान्यथाभूतत्वात्।"

i.e., "Similarly, in the *manda* operation of the planets, Mars, etc., too, having obtained the (*manda*) *karṇa* in the manner stated above, multiply the *manda* epicycle by that and divide (the product) by the radius : the result is the (*manda*) epicycle in the *karṇavṛtta* (i.e., at the distance of the *mandakarṇa*). Determining from that the *bhujāphala* and the *koṭiphala* again, in the manner stated before, obtain the *mandakarṇa*. Perform this process (again and again) until there is no difference in the result (i.e., until the nearest approximation for the true *manda* epicycle is obtained)."

"Conversion of the false *manda* epicycle into the true *manda* epicycle is done by the (following) proportion : If at the distance of the radius we get the measure of the (false) epicycle, what shall we get at the distance of the (*manda*) *karṇa* ? The result is the *manda* epicycle at the distance of the (*manda*) *karṇa*. Iteration of the true *manda* epicycle is done because the (*manda*) *karṇa* is of a different nature (i.e. because the *mandakarṇa* is obtained by iteration)."

From what has been stated above it is evident that the *manda* epicycles stated in the works on Hindu astronomy correspond to the radius of the deferent and are false, whereas the true *manda* epicycles which are derived therefrom by formula (3) above correspond to the true distance (true *mandakarṇa*) of the planet and are the actual epicycles on which the planet (in the case of the Sun and the Moon) or the true-mean planet (in the case of the planets Mars, etc.) moves.

Therefore, if we use the tabulated *manda* epicycle, we shall get

$$bhuj\bar{a}phala = \frac{\text{tabulated } manda \text{ epicycle} \times R \sin m}{80}, \tag{4}$$

where m is the planet's mean *mandakendra* (reduced to *bhuja*), the tabulated *manda* epicycle being abraded by $4\frac{1}{2}$ as is usual in the school of Āryabhaṭa I.

Since the tabulated *manda* epicycle corresponds to the radius of the deferent, there is absence of the hypotenuse-proportion and we have

$$R \sin \text{(equation of centre)} = bhuj\bar{a}phala$$

$$= \frac{\text{tabulated } manda \text{ epicycle} \times R \sin m}{80},$$

which is the formula used in the school of Āryabhaṭa I.

If we choose to use the true *manda* epicycle, we shall get

$$\text{true } bhuj\bar{a}phala = \frac{\text{true } manda \text{ epicycle} \times R \sin m}{80},$$

and since this true *bhujāphala* corresponds to the true *mandakarṇa*, therefore, applying the hypotenuse-proportion, we have

$$R \sin (\text{equation of centre}) = \frac{\text{true } bhuj\bar{a}phala \times R}{H}, \tag{5}$$

where H is the true *mandakarṇa* (obtained by iteration).

Substituting the value of true *bhujāphala* and making use of formula (3), equation (5) reduces to

$$R \sin (\text{equation of centre}) = \frac{\text{tabulated } manda \text{ epicycle} \times R \sin m}{80}.$$

But this result is the same as (4) which was obtained without the use of the hypotenuse-proportion. This explains why in the school of Āryabhaṭa I, the *mandakarṇa* (true hypotenuse) is not used in the computation of the equation of the centre under the epicyclic theory.

3. Views of Astronomers of the School of Āryabhaṭa I

3.1. *Bhāskara I (629)*

In his commentary on the *Āryabhaṭīya*, Bhāskara I, the greatest authority on Āryabhaṭa I, raises the question as to why the hypotenuse was used in finding the *śīghraphala* but was not used in finding the *mandaphala* (i.e. equation of centre) and answers it. Writes he :[5]

अत्र शीघ्रफलं व्यासार्धेन संगुणय्य तदुत्पन्नकर्णेन भागलब्धं फलं धनमृणं वा।····अनेनाथ मन्दोच्चफलमेवं कस्मान्न क्रियते ? उच्यते—क्रियमाणेऽपि तावदेव तत्फलं भवतीति न क्रियते। कुतः ? मन्दोच्चकर्णोऽविशिष्यते। तत्र चाविशेषितेन फलेन व्यासार्धं संगुणय्य कर्णेन भागे हृते पूर्वमानीतमेव फलं भवतीति। अथ किमिति शीघ्रोच्चकर्णो नाविशिष्यते ? अभावादविशेषकर्मणः।

i.e. "Here the *śīghra* (*bhujā*)*phala* is got multiplied by the radius and divided by the *śīghrakarṇa* and the quotient (obtained) is added or subtracted (in the manner prescribed).......

Question : How is it that the *manda* (*bhujā*)*phala* is not operated upon in this way (i.e. why is the *mandabhujāphala* not multiplied by the radius and divided by the *mandakarṇa*) ?

Answer : Even if it is done, the same result is obtained as was obtained before ; that is why it is not done.

Question : How ?

Answer : The *mandakarṇa* is iterated. Therefore when we multiply the iterated (*mandabhujā*)*phala* (i.e. true *mandabhujā-phala*) by the radius and divide by the (true) *mandakarṇa*, we obtain the same result as was obtained before.

Question : Now, how is it that the *śīghrakarṇa* is not iterated ?

Answer : This is because the process of iteration does not exist there."

3.2. *Govinda Svāmi* (*c. 800-850*)

Govinda Svāmi, who is another important exponent of the school of Āryabhaṭa I, raises the same question and answers it in the same way. Writes he :[6]

कथं पुनरिदं मन्दफलं प्रतिमण्डले न प्रमीयते ? कृतेऽपि पुनस्तावदेवेति। कथम् ? मन्दोच्च-कर्णस्य तावदविशेष उक्तः। अविशिष्टात् फलाद् व्यासार्धहतात् कर्णेन (हृतात्) पूर्वानीतमेव फलं लभ्यत इति। किमिति शीघ्रकर्णो नाविशिष्यते ? अविशेषाभावात्।

"*Question* : How is it that the *manda* (*bhujā*)*phala* is not measured in the *manda* eccentric (i.e. How is it that the *mandabhujāphala* is not calculated at the distance of the planet's *mandakarṇa*) ?

Answer : Even if that is done, the same result is got.

Question : How ?

Answer : Because iteration of the *mandakarṇa* is prescribed. So when the iterated (i.e. true) *bhujāphala* is multiplied by the radius and divided by the (true *manda*) *karṇa*, the same result is obtained as was obtained before.

Question : How is it that the *śīghrakarṇa* is not iterated ?

Answer : Because there is absence of iteration."

3.3. *Parameśvara* (*1430*)

So also writes the celebrated Parameśvara :[7]

मन्दस्फुटे तु कर्णस्याविशेषितत्वान्मंदफलमपि अविशेषितं भवति। अविशिष्टात् पुनर्मन्दफलाद् व्यासार्धताडिताद् अविशिष्टेन कर्णेन लब्धं प्रथमानीतमेव भुजाफलं भवति।

i.e. "In the case of the *manda* correction, the (*manda*) *karṇa* being subjected to iteration the *manda* (*bhujā*)*phala* is also got iterated (in the process). So, the iterated *manda* (*bhujā*)*phala* being multiplied by the radius and divided by the iterated *mandakarṇa*, the result obtained is the same *bhujāphala* as was obtained in the beginning."

3.4. *Nīlakaṇṭha* (*c. 1500*)

Nīlakaṇṭha, author of the *Mahābhāṣya* on the *Āryabhaṭīya* and an eminent authority on Āryabhaṭa I, says the same thing in his *Mahābhāṣya* :[8]

पूर्वं तु केवलमन्त्यफलमविशिष्टेन कर्णेन हत्वा व्यासार्धहृतमेवाविशिष्टमन्त्यफलम्। तदेव पुनर्व्यासार्धेन हत्वा कर्णेन हृतं पूर्वतुल्यमेव स्याद्, यत उभयोस्त्रैराशिककर्मणोर्मिथो वैपरीत्यं स्यात्। एतदुक्तं महाभास्करीयभाष्ये—'कृतेऽपि पुनस्तावदेवे'ति। तस्मान्मन्दकर्मणि भुजाफलं न कर्णसाध्यम्। केवलमेव मध्यमे संस्कार्यम्। शीघ्रे तु कर्णवशाद् उच्चनीचवृत्तस्य वृद्धिह्रासाभावात् सकृदेव कर्णः कार्यः। भुजाफलमपि व्यासार्धेन हत्वा कर्णेन हृतमेव चापीकार्यम्।

i.e. "Earlier, the iterated *antyaphala* (= radius of epicycle) was obtained by multiplying the uniterated *antyaphala* by the iterated hypotenuse and dividing (the product) by the radius. The same (i.e. iterated *antyaphala*) having been multiplied by the radius and divided by the (iterated) hypotenuse yields the same result as the earlier one, because the two processes of "the rule of three" are mutually reverse. The same has been stated in the *Mahābhāskarīyabhāṣya* (i.e. in the commentary on the *Mahābhāskarīya* by Govinda Svāmi) : 'Even if that is done, the same result is got.' So in the *manda* operation, the *bhujāphala* is not to be determined by the use of the (*manda*) *karṇa* ; the (uniterated) *bhujāphala* itself should be applied to the mean (longitude of the) planet. In the *śīghra* operation, since the *śīghra* epicycle does not vary with the hypotenuse, the *karṇa* should be calculated only once (i.e., the process of iteration should not be used). The *bhujāphala*, too, should be multiplied by the radius, (the product obtained) divided by the hypotenuse, and (the resulting quotient) should be reduced to arc."

What is meant is that if we first find the true *antyaphala* (radius of the true *manda* epicycle) by the formula

$$\text{true } antyaphala = \frac{\text{radius of uniterated } manda \text{ epicycle} \times H}{R},$$

and then apply the hypotenuse proportion, we shall again get the radius of the uniterated *manda* epicycle with which we started. So the final result, viz.

$$R \sin \text{(equation of centre)} = \frac{\text{radius of uniterated } manda \text{ epicycle} \times R \sin m}{R},$$

may be obtained directly without finding the radius of the iterated *manda* epicycle and then applying the hypotenuse-proportion.

3.5. *Sūryadeva Yajvā* (*b. 1191*)

The same thing has been stated in a slightly different way by the commentator Sūryadeva, who writes :[9]

अत्राचार्येण कक्ष्यामण्डलकलाभिर्मन्दनीचोच्चवृत्तानि पठितानि। अतस्तद्गतैव ज्या काष्ठीकृता कक्ष्यामण्डलकलासाम्यात्तत्स्थे मध्यग्रहे संस्क्रियते। कर्णानयने तु तद्वृत्तपरिणामाय त्रैराशिकं कृत्वा

अविशेषेण कर्णः कर्तव्यः। शीघ्रवृत्तानि तु प्रतिमण्डलस्थान्येवाचार्येण पठितानि। अतः फलज्यायाः कक्ष्यामण्डलपरिणामार्थं त्रैराशिकं—कर्णस्येयं ज्या व्यासार्धस्य केति ? लब्धा फलज्या चापीकृता कक्ष्यामण्डलसदृशी मन्द(स्पष्ट)ग्रहे संस्क्रियते। कर्णानयनं तु सकृत्कर्मणैव कार्यम्।

i.e., "Here the *Ācārya* (viz. Ācārya Āryabhaṭa I) has stated the *manda* epicycles in terms of the minutes of the deferent. So the (*mandabhujāphala*) *jyā* which pertains to that (deferent) when reduced to arc, its minutes being equivalent to the minutes of the deferent, is applied (positively or negatively as the case may be) to (the longitude of) the mean planet situated there (on the deferent). In finding the (*manda*) *karṇa*, however, one should, having applied the rule of three in order to reduce the *manda* epicycle to the circle of the (*mandakarṇa*), obtain the (true *manda*) *karṇa* by the process of iteration. The *śīghra* epicycles, on the other hand, have been stated by the *Ācārya* for the positions of the planets on the (true) eccentric. So, in order to reduce the (*śīghrabhuja*) *phalajyā* to the concentric, one has to apply the proportion : If this (*śīghrabhujāphala*)*jyā* corresponds to the (*śīghra*) *karṇa*, what *jyā* would correspond to the radius (of the concentric) ? The resulting (*śīghra*)*phalajyā* reduced to arc, being identical with (the arc of) the concentric, is applied to (the longitude of) the true-mean planet. The determination of the (*śīghra*) *karṇa*, however, is to be made by a single application of the rule (and not by the process of iteration)."

3.6. *Putumana Somayâji* (*1732*)

A glaring example of the fact that the astronomers of the school of Āryabhaṭa I regarded the *manda* epicycles as corresponding to the mean distances of the planets and the *śīghra* epicycles as corresponding to the actual distances of the planets is provided by the following rule occurring in the *Karaṇa-paddhati* (vii.27) of Putumana Somayāji, a notable exponent of the Āryabhaṭa school :

Let $4\frac{1}{2} \times e$ be the periphery of a planet's *manda* epicycle at the beginning of the odd anomalistic quadrant and $4\frac{1}{2} \times e'$ the periphery of a planet's *śīghra* epicycle at the beginning of the odd anomalistic quadrant. Then, the planet being at its *mandocca* (apogee),

$$mandakarṇa = \frac{80 \times R}{80 - e},$$

and, the planet being at its *mandanīca* (perigee),

$$mandakarṇa = \frac{80 \times R}{80 + e}.$$

On the other hand, the planet being at its *śīghrocca*,

$$śīghrakarṇa = \frac{(80 + e') \times R}{80},$$

and, the planet being at its *śīghranīca*,

$$śīghrakarṇa = \frac{(80 - e') \times R}{80}.$$

4. Views of Astronomers of Other Schools

4.1. *Brahmagupta's view : Caturvedācārya Pṛthūdaka's disagreement : Bhāskara II's judgement.*

The astronomers of the Brahma school also use false *manda* epicycles and likewise they do not make use of the hypotenuse in the computation of the equation of the centre under the epicyclic theory. Brahmagupta (628), the author of the *Brāhmasphuṭasiddhānta* and the main exponent of this school, explains the reason for not using the hypotenuse in finding the *mandaphala* as follows :[10]

त्रिज्याभक्तः परिधिः कर्णगुणो बाहुकोटिगुणकारः ।
असकृन्मान्दे तत्फलमाद्यसमं नात्र कर्णोऽस्मात् ॥

i.e. "In the *manda* operation (i.e. in finding the *mandaphala*), the *manda* epicycle divided by the radius and multiplied by the hypotenuse is made the multiplier of the *bāhu(jyā)* and the *koṭi(jyā)* in every round of the process of iteration. Since the *mandaphala* obtained in this way is equivalent to the *bhujāphala* obtained in the beginning, therefore the hypotenuse-proportion is not used here (in finding the *mandaphala*)."

This is the same explanation as was given by the astronomers of the school of Āryabhaṭa I.

Caturvedācārya Pṛthūdaka (864), on the other hand, was of the opinion that the hypotenuse-proportion was not applied in finding the equation of the centre because it did not produce any material difference in the result. He has therefore remarked :"[11]

अतः स्वल्पान्तरत्वात् कर्णो मन्दकर्मणि न कार्यः इति ।

"So, there being little difference in the result, the hypotenuse-proportion should not be used in finding the *mandaphala*."

The celebrated Bhāskara II (1150), the author of the *Siddhāntaśiromaṇi*, has examined the views of both Brahmagupta and Caturvedācārya Pṛthūdaka and has given his verdict in favour of Brahmagupta's view. Writes he :[12]

यो मन्दपरिधिः पाठपठितः स त्रिज्यापरिणतः । अतोऽसौ कर्णव्यासार्धे परिणाम्यते । ततोऽनुपातः । यदि त्रिज्यावृत्तेऽयं परिधिस्तदा कर्णवृत्ते क इति । अत्र परिधेः कर्णो गुणस्त्रिज्या हरः । एवं स्फुटपरिधिस्तेन दोर्ज्या गुण्या भांशैर्भाज्या । ततस्त्रिज्यया गुण्या कर्णेन भाज्या । एवंसति त्रिज्यातुल्ययोः कर्णतुल्ययोश्च गुणहरयोस्तुल्यत्वान्नाशे कृते पूर्वफलतुल्यमेव फलमागच्छतीति ब्रह्मगुप्तमतम् । अथ यद्येवं परिधेः कर्णेन स्फुटत्वं तर्हि किं शीघ्रकर्मणि न कृतमित्याशङ्क्य चतुर्वेद आह । ब्रह्मगुप्तेनान्येषां प्रतारणपरमिदमुक्तमिति । तदसत् । चले कर्मणीत्थं किं न कृतमिति नाशङ्कनीयम् । यतः फलवासना विचित्रा । शुक्रस्यान्यथा परिधेः स्फुटत्वं भौमस्यान्यथा तथा किं न बुधादीनामिति नाशङ्क्यम् । अतो ब्रह्मोक्तिरत्र सुन्दरी ।

i.e. "The *manda* epicycle which has been stated in the text is that reduced to the radius of the deferent. So it is transformed to correspond to the radius equal to the hypotenuse (of the planet). For that the proportion is : If in the radius-circle we have this epicycle, what shall we have in the hypotenuse circle ? Here the epicycle has the hypotenuse for its multiplier and the radius for its divisor. Thus is obtained the true epicycle. The *bhujajyā* is multiplied by that and divided by 360. That is then multiplied by the radius and divided by the hypotenuse. This being the case, radius and hypotenuse both occur as multiplier and also as divisor and so they being cancelled the result obtained is the same as before : this is the opinion of Brahmagupta. If the epicycle is to be corrected in this way by the use of the hypotenuse, why has the same not been done in the *śīghra* operation ? With this doubt in mind, Caturveda has said : "Brahmagupta has said so in order to deceive and mislead others." That is not true. Why has that not been done in the *śīghra* operation, is not to be questioned, because the rationales of the *manda* and *śīghra* corrections are different. Correction of Venus' epicycle is different and that for Mars' epicycle different ; why is that for the epicycles of Mercury etc. not the same, is not to be questioned. Hence what Brahmagupta has said here is right."

4.2. *Śrīpati (c. 1039)*

Śrīpati, author of the *Siddhāntaśekhara*, has expressed the same opinion as Brahmagupta has done. He has written :[13]

त्रिज्याहृतः श्रुतिगुणः परिधिर्यतो दोः-
कोट्योर्गुणो मृदुफलानयनेऽसकृत्स्यात् ।
स्यान्मन्दमाद्यसममेव फलं ततश्च
कर्णः कृतो न मृदुकर्मणि तन्त्रकारैः ॥

i.e. "Since in the determination of the *mandaphala* the epicycle multiplied by the hypotenuse and divided by the radius is repeatedly made the multiplier of the *bhuja* (*jyā*) and the *koṭi*(*jyā*), and since the *mandaphala* obtained in this way is equal to the *bhujāphala* obtained in the beginning, therefore the hypotenuse-proportion has not been applied in the *manda* operation by the authors of the astronomical *tantras*."

4.3. *Āditya Pratāpa*

The same view was held by the author of the *Ādityapratāpasiddhānta*, whose words are :[14]

भवेत्कक्षाभवो मन्दपरिधिः प्रतिमण्डले ।
मृदुकर्णगुणः स्पष्टः कक्षाव्यासदलोद्धृतः ॥
तद्बाहुकोटितः प्राग्वत्कर्णः साध्योऽसकृत् स्फुटः ।
तेन बाहुफलं भक्तं कक्षाव्यासार्धसङ्गुणम् ॥

भवेन्मन्दफलं मध्यपरिध्युत्पन्नसम्मितम् ।
यत्नेन न कृतः कर्णः फलार्थं मन्दकर्मणि ॥

i.e., "The *manda* epicycle corresponding to (the radius of) the orbit (concentric), when multiplied by the *mandakarṇa* and divided by the semi-diameter of the orbit (concentric) becomes true and corresponds to (the distance of the planet on) the eccentric. With the help of that (true epicycle), the *bāhu(jyā)*, and the *koṭi(jyā)* should be obtained the true *karṇa* by proceeding as before and by iterating the process. Since the (true) *bāhuphala* divided by that (true *karṇa*) and multiplied by the semi-diameter of the orbit yields the same *mandaphala* as is obtained from the mean epicycle (without the use of the hypotenuse-proportion), therefore use of the hypotenuse-(proportion) has not been made for finding the *mandaphala* in the *manda* operation."

4.4. *The Sūryasiddhānta school.*

The method prescribed in the *Sūryasiddhānta* for finding the equation of the centre is exactly the same as given by the exponents of the schools of Āryabhaṭa I and Brahmagupta and there is no use of the hypotenuse-proportion. The author of the *Sūryasiddhānta* has not even taken the trouble of finding the *manda* hypotenuse. So it may be presumed that the views of the author of the *Sūryasiddhānta* on the omission of the use of the hypotenuse in finding the equation of the centre were similar to those obtaining in the schools of Āryabhaṭa I and Brahmagupta.

5. Conclusion

From what has been said above it is clear that the hypotenuse has not been used in Hindu astronomy in the computation of the equation of the centre under the epicyclic theory. It is also obvious that with the single exception of Caturvedācārya Pṛthūdaka all the Hindu astronomers are unanimous in their views regarding the cause of omission of the hypotenuse. According to all of them the *manda* epicycles stated in the works on Hindu astronomy correspond to the radius of the planet's mean orbit and are therefore false.

Since the *manda* epicycle stated in the Hindu works corresponded to the radius of the planet's mean orbit, the true *manda* epicycle corresponding to the planet's true distance (in the case of the Sun and Moon) or true-mean distance (in the case of the planets Mars, etc.) was obtained by the process of iteration. The planet's true or true-mean distance (*mandakarṇa*) was also likewise obtained by the process of iteration.

Direct methods for obtaining the true *mandakarṇa* or true *manda* epicycle were also known to later astronomers. Mādhava (c. 1340-1425) is said to have given the following formula for the true *mandakarṇa* :[15]

$$\text{true } mandakar\d{n}a \text{ (or iterated } mandakar\d{n}a) = \frac{R^2}{\sqrt{R^2-(bhuj\bar{a}phala)^2} \mp ko\d{t}iphala},$$

$\sim$ or $+$ sign being taken according as the planet is in the half orbit beginning with the anomalistic sign Capricorn or in that beginning with the anomalistic sign Cancer.

The following alternative formula is attributed by Nilakaṇṭha (*c*.1500) to his teacher (Dāmodara) :[16]

true *mandakarṇa* (or iterated *mandakarṇa*)

$$= \frac{R^2}{\sqrt{(\text{true } koṭijyā \,\tilde{+}\, antyaphalajyā)^2 + (\text{true } bhujajyā)^2}}$$

$\sim$ or $+$ sign being taken according as the planet is in the half orbit beginning with the anomalistic sign Capricorn or in that beginning with the anomalistic sign Cancer.

The following alternative formula occurs in the *Karaṇa-paddhati* (vii. 17, 18, 20(ii)) of Putumana Somayāji :

true *mandakarṇa* (or iterated *mandakarṇa*)

$$= \frac{R^2}{\sqrt{(R \pm koṭiphala)^2 + (bhujāphala)^2}},$$

$+$ or $-$ sign being taken according as the planet is in the half-orbit beginning with the anomalistic sign Cancer or in that beginning with the anomalistic sign Capricorn.[17]

One can easily see that each of these formulae gives an exact expression for the iterated *mandakarṇa*.

6. Use of Hypotenuse Under the Eccentric Theory Indispensable

The problem of finding the *spaṣṭabhuja* (true *manda* anomaly reduced to *bhuja*) under the eccentric theory is quite different. Here one has to take the planet on its true *manda* eccentric and has to apply the proportion : "When corresponding to the radius vector equal to the iterated *mandakarṇa* one gets the *madhyama bhujajyā*, what shall one get corresponding to the radius R of the concentric ?" The result is the Rsine of the *spaṣṭabhuja* equal to

$$\frac{(madhyama\ bhujajyā) \times R}{H},$$

where H is the true (or iterated) *mandakarṇa*.

It must be noted that the planet moves on the true *manda* eccentric whose centre is displaced from the Earth's centre by an amount equal to the radius of the true *manda* epicycle. Bhāskara I writes :[18]

परिधिचालनाप्रयोगेण स्फुटीकृतपरिधिना व्यासार्धं संगुणय्याशीत्या भागलब्धं प्रतिमण्डलभूविवरम् ।

i.e. "Multiply the radius by the epicycle rectified by the process of iteration and divide by 80 : the quotient obtained is the distance between the centres of the eccentric and the Earth."

This shows that the Hindu epicyclic theory in which the equation of the the centre is obtained directly without the use of the hypotenuse-proportion is much simpler than the Hindu eccentric theory in which the use of the iterated hypotenuse is indispensable. It is for this reason that the use of the epicyclic theory has been more popular in Hindu astronomy than the eccentric theory. The *Sūryasiddhānta* and other works, which have avoided finding the iterated hypotenuse, have dispensed with the eccentric theory altogether.

7. Exceptions. Use of True Manda Epicycle

Munīśvara (1646) and Kamalākara (1658), who claim to be the followers of the *Siddhāntaśiromaṇi* of Bhāskara II and the *Sūryasiddhānta* respectively, are perhaps the only two Hindu astronomers who, disregarding the general trend of Hindu astronomy, have stated the dimensions of the true *manda* epicycles in their works and have likewise used the hypotenuse-proportion in finding the equation of the centre under the epicyclic theory. The formula for the equation of the centre given by them is :[19]

$$R \sin\,(\text{equation of centre}) = \frac{bhuj\bar{a}phala \times R}{H}, \tag{6}$$

where H is the *mandakarṇa*. Since they have used the true *manda* epicycle, they have obtained the *mandakarṇa* directly without making use of iteration ; this is as it should be.

It is noteworthy that although Kamalākara makes use of the true *manda* epicycle and uses formula (6) above, he does not forget to record the fact that the *bhujāphala* obtained directly by the use of the *manda* epicycle corresponding to the radius of the planet's mean orbit yields the same result as formula (6) above. Writes he :[20]

त्रिज्याहतः कर्णहृतः कृतश्चेद्
यथोक्त आद्यः परिधिः स्फुटः स्यात् ।
तत्साधितं दोःफलचापमेव
फलं भवेद्वोक्तफलेन तुल्यम् ॥

i.e., "The true (*manda*) epicycle as stated earlier when multiplied by the radius and divided by the hypotenuse becomes corrected (i.e. corresponds to the radius of the planet's mean orbit). The arc corresponding to the *bhujāphala* computed therefrom yields the equation of centre which is equal to that stated before."

References

[1] *Indian Journal of History of Science*, **4**, (1) & (2), pp. 126-134.

[2] *Mahābhāskarīya*, edited by T. S. Kuppanna Shastri, p. 224, lines 15-17.

[3] *Śiṣyadhīvṛddhida*, I, iii. 17.

[4] Bhāskara II 's comm. on *Śiṣyadhīvṛddhida*, I, iii. 17,

[5] Bhāskara I 's comm. on *Āryabhaṭīya*, iii, 22.

[6] *Mahābhāskarīya*, edited by T. S. Kuppanna Shastri, p. 224, lines 1-4.

[7] *Mahābhāskarīya*, ed. T. S. Kuppanna Shastri, p. 223, line 22, p. 224, lines 12-13.

[8] Nīlakaṇṭha's comm. on *Āryabhaṭīya*, iii. 17-21, p. 43, lines 4-10.

[9] Sūryadeva's comm. on *Āryabhaṭīya*, iii. 24.

[10] *Brāhmasphuṭasiddhānta*, *golādhyāya*, 29.

[11] Pṛthūdaka's comm. on *Brāhmasphuṭasiddhānta*, *golādhyāya*, 29.

[12] *Siddhāntaśiromaṇi golādhyāya*, *Chedyakādhikāra*, 36-37, comm.

[13] *Siddhāntaśekhara*, xvi, 24.

[14] Āmarāja's comm. on *Khaṇḍakhādyaka*, i.16, p. 33.

[15] Nīlakaṇṭha's comm, on *Āryabhaṭīya*, iii. 17-21, p. 47. Also see *Tantrasaṅgraha*, ii, 44.

[16] Nīlakaṇṭha's comm. on *Āryabhaṭīya*, iii. 17-21, p. 48. Also see *Tantrasaṅgraha*, i. 46–47.

[17] The *bhujāphala* and *koṭiphala* used in this formula are those derived from true *bhujajyā* and true *koṭijyā*. This formula was known to Mādhava and Nīlakaṇṭha also. See Nīlakaṇṭha's commentary on *Āryabhaṭīya*, iii. 17-21, pp. 48-49 and *Tantra-saṅgraha*, i. 51.

[18] Nīlakaṇṭha's comm, on *Āryabhaṭīya*, iii. 21.

[19] *Siddhāntasārvabhauma*, ii. 124 (i) ; *Siddhāntatatvaviveka*, ii. 207(i).

[20] *Siddhāntatatvaviveka*, ii. 208.