TRIGONOMETRICAL SERIES IN THE *KARAṆAPADDHATI* AND THE PROBABLE DATE OF THE TEXT

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The paper discusses the trigonometrical series for π, sine, cosine and tan functions contained in the *Karaṇapaddhati*. The original texts and their translations are given. As to its date C. M. Whish attempted to fix it in the beginning of the eighteenth century. Whish's derivation is untenable and it is shown that the text is likely to be contemporaneous with, or even to antidate, the *Tantrasaṃgraha*, of Nīlakaṇṭha Somasuttvan (A.D 1465–1545).

1. INTRODUCTION

The *Karaṇapaddhati* is an important astronomical work in ten chapters by an unknown Kerala astronomer of uncertain date. Only this much is known about the author that he was a Brāhmin who took his abode in the village of Śivapura.[1] The text in Devanāgarī script is published by K. Sambaśiva Śāstrī,[2] but this edition misses the opening verse.[3] Apart from the usual elements and formulae characteristic of Hindu astronomy, the work gives, in the sixth chapter, trigonometrical π, sine, cosine and tan series. The mathematical importance of the text was first pointed out by C. M. Whish[4] who discussed π and tan functions given in the text and attempted to fix its date to the beginning of the eighteenth century A.D. In this connection Whish also drew attention to three texts, e.g. *Tantrasaṃgraha*, *Yuktibhāṣā* and *Sadratnamālā*, containing similar trigonometrical series. Some of these series have been studied by C. T. Rājagopal *et al.*[5] and T. A. Saraswati.[6] It is proposed to consider here all the four trigonometrical series given in the *Karaṇapaddhati*, including the sine and cosine not discussed by Whish, as also the question of date of the text.

2.1. π SERIES

This series has been discussed in chapter 6 of the text and a number of series have been given. The opening verse runs as follows:

vyāsāccaturghnād bahuśaḥ pṛthaksthāt tripañcasaptādyayugāhṛtāni |
vyāse caturghne kramaśastvṛṇaṃ svaṃ kurjāt tadā syāt paridhiḥ susūkṣmaḥ ||
(ch. 6, 1)

'Four times of the diameter is to be divided separately by each of the odd integers 3, 5, 7 . . . ; every quotient whose order is even is taken away from the one preceding it. Combined result of all such small operations, when subtracted from four times the diameter, gives the value of the circumference with progressively greater accuracy.'

If C be the circumference and D the diameter, the rule may be expressed as:

$$C = 4D - 4D\left(\frac{1}{3}-\frac{1}{5}\right) - 4D\left(\frac{1}{7}-\frac{1}{9}\right) - \ldots$$

$$\text{or,} \qquad \pi/4 = \left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\right)$$

The next verse gives the series in a different form as follows:

vyāsād vanasaṅguṇitāt pṛthagāptaṃ tryādyayug-vimūlaghanaiḥ |
triguṇavyāse svamṛṇaṃ kramaśaḥ kṛtvāpi paridhirāneyaḥ || (ch. 6, 2)

'Four times the diameter is divided separately by the cubes of the odd integers from 3 onwards, diminished by these integers themselves. The quotients thus obtained are alternately added to and subtracted from thrice the diameter. The result is the circumference.'

Thus,

$$C = 3D + 4D\left\{\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\ldots\right\}$$

$$\text{or,} \qquad \pi = 3+4\left\{\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\ldots\right\}$$

Another series for π is given as follows:

vargairyujāṃ vā dviguṇairnirekairvargīkṛtair-varjitayugmavargaiḥ |
vyāsaṃ ca ṣaḍghnaṃ vibhajet phalaṃ svaṃ vyāse trinighne paridhistadā syāt ||
(ch. 6, 4)

'Six times the diameter is divided separately by the square of twice the squares of even integers (2, 4, 6 . . .) minus 1, diminished by the squares of the even integers themselves. The sum of the resulting quotients increased by thrice the diameter is the circumference.'

Thus,

$$C = 3D+6D\left\{\frac{1}{(2\,.\,2^2-1)^2-2^2}+\frac{1}{(2\,.\,4^2-1)^2-4^2}+\frac{1}{(2\,.\,6^2-1)^2-6^2}+\ldots\right\}$$

$$\text{or,} \quad \pi = 3+6\left\{\frac{1}{(2\,.\,2^2-1)^2-2^2}+\frac{1}{(2\,.\,4^2-1)^2-4^2}+\frac{1}{(2\,.\,6^2-1)^2-6^2}+\ldots\right\}$$

$$\text{or,} \quad \pi = 3+6\left\{\frac{1}{1\,.\,3\,.\,3\,.\,5}+\frac{1}{3\,.\,5\,.\,7\,.\,9}+\frac{1}{5\,.\,7\,.\,11\,.\,13}+\frac{1}{7\,.\,9\,.\,15\,.\,17}+\ldots\right\}$$

2.2. Sine and Cosine Series

The rules for the expression of sine and cosine functions in the form of series are given in the same chapter:

cāpācca tattat phalato'pi tadvat cāpāhatāddvyādihatat trimaurvyā |
labdhāni yugmāni phalānyadhodhaḥ cāpādayugmāni ca vistarārdhāt ||
vinyasya coparyupari tyajet tat śeṣau bhūjākoṭiguṇau bhavetām |

(ch. 6, 12–13½)

'The arc repeatedly multiplied by itself (any number of times) is multiplied by the arc. The result is divided by the product of 2, 3 . . . , etc. (up to the same number), and the radius is repeatedly multiplied by itself (the same number of times). The quotients (thus obtained) corresponding to the even numbers (up to which the aforesaid multiplication is repeatedly done) are set one below the other (in one column) and likewise those corresponding to the odd numbers (in another column). From the first term is subtracted the term immediately below and so on, and the remainders for the even column are subtracted from the are and those for the odd column from the radius. The results are the *bhūjajyā* and *koṭijyā* respectively.'

To explain the rule in the circle (Fig. 1),

$SP =$ *cāpa*, s
$AP =$ *trimaurvā* or radius, r
$PM =$ *bhujajyā*, $r \sin \theta$
$AM =$ *koṭijyā*, $r \cos \theta$

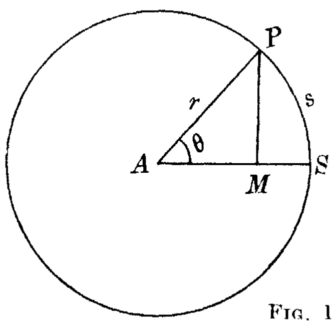

Fig. 1

If n be the number of times up to which the arc s be multiplied by itself, the first part of the rule gives the quotient as:

$$q_n = \frac{s^n . s}{(2 . 3 \ldots n \text{ terms}) \times r^n}$$
$$= \frac{s^{n+1}}{(n+1)! \; r^n}.$$

In the next step, the even quotients, $q_2, q_4, q_6 \ldots$ are set down in a column and differences $(q_2-q_4), (q_6-q_8) \ldots$, etc., are determined. Similarly by arranging the odd quotients in another column, the differences $(q_1-q_3), (q_5-q_7) \ldots$, etc., are obtained. Then *bhujajyā* is given by

$$r \sin \theta = s-(q_2-q_4)-(q_6-q_8) - \ldots$$
$$= s-\left(\frac{s^3}{3! \; r^2} - \frac{s^5}{5! \; r^4}\right)-\left(\frac{s^7}{7! \; r^6} - \frac{s^9}{9! \; r^8}\right)- \ldots$$
$$\text{or, } \sin \theta = \frac{s}{r} - \frac{1}{3!}\left(\frac{s}{r}\right)^3 + \frac{1}{5!}\left(\frac{s}{r}\right)^5 - \frac{1}{7!}\left(\frac{s}{r}\right)^7 + \frac{1}{9!}\left(\frac{s}{r}\right)^9 - \ldots$$

Koṭijyā is given by

$$r \cos \theta = r-(q_1-q_3)-(q_5-q_7)-\ldots$$

$$= r-\frac{s^2}{2!\,r}+\frac{s^4}{4!\,r^3}-\frac{s^6}{6!\,r^5}+\frac{s^8}{8!\,r^7}-\ldots$$

$$\text{or, } \cos \theta = 1-\frac{1}{2!}\left(\frac{s}{r}\right)^2+\frac{1}{4!}\left(\frac{s}{r}\right)^4-\frac{1}{6!}\left(\frac{s}{r}\right)^6+\frac{1}{8!}\left(\frac{s}{r}\right)^8-\ldots$$

From these, the modern forms can be readily obtained by putting $s = r\theta$ for small values of s or θ as follows:

$$\sin \theta = \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots$$

$$\text{and } \cos \theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots$$

In these forms, the series appeared for the first time in Europe in a letter which Newton sent to Oldenburg in 1676.[7] The same series, as well known, were derived later as a corollary to De Moivre's Theorem.

2.3. Tan Series

For the tan series, the verse runs as follows:

vyāsārdhena hatādabhiṣṭaguṇataḥ koṭyāptamādyaṃ phalaṃ
jyāvargeṇa vinighnamādimaphalaṃ tattatphalaṃ cāharet |
kṛtyā koṭiguṇasya tatra tu phaleṣvekatripañcādibhir-
bhakteṣvojayutaistajet samajutiṃ jīvādhanuśiśaṣyate || (ch. 6, 18)

'Ordinate of the arc (*dhanuṣ*) is to be multiplied by the semi-diameter (*vyāsārdha*) and is divided by the abscissa (*koṭi*). This is the first term. Thus the result obtained is multiplied by the square of the ordinate (*jyā*) and is divided by the square of the abscissa. This is the second term. This process is repeated. The successive terms are divided by the odd integers 1, 3, 5, ... Now, the terms of odd order are added and the terms of even order are subtracted from the preceding, the circumference will be obtained.'

Abscissa* $AM \geqslant$ Ordinate PM, i.e. SAP $\leqslant 45°$ (Fig. 1), then according to the above translation, it can be written as:

$$\text{arc } SP = AP\left[\frac{PM}{AM}-\frac{1}{3}\frac{PM^3}{AM^3}+\frac{1}{5}\frac{PM^5}{AM^5}-\ldots\right]$$

$$\text{or, } s = r\left[\frac{r \sin \theta}{r \cos \theta}-\frac{1}{3}\left(\frac{r \sin \theta}{r \cos \theta}\right)^3+\frac{1}{5}\left(\frac{r \sin \theta}{r \cos \theta}\right)^5-\ldots\right.$$

$$\text{Hence, } \theta = \tan \theta-\frac{1}{3}\tan^3 \theta+\frac{1}{5}\tan^5 \theta-\ldots \text{ when } s = r\theta.$$

* This restriction is put by the commentator.

Karaṇapaddhati gives no proof of these series. These series also appear in the *Tantrasaṃgraha* to which Whish already drew attention.[8] *Yuktibhāṣā*, an exposition of Nīlakaṇṭha's *Tantrasaṃgraha*, gives proofs to all of these series.[9]

In the general history of mathematics, the π and the tan series are associated with the name of Gregory (A.D. 1638–1675).[10]

3. The Date of *Karaṇapaddhati*

Charles M. Whish, in his effort to fix the date of *Karaṇapaddhati*, quoted the concluding verse as follows:

iti śivapuranāma-grāmajaḥ ko'pi yajvā
kimapi karaṇapaddhatyāhvayaṃ tantrarūpam |
vyadhita gaṇitametat samyagālokya santaḥ
kathitamiha vidantaḥ santu santoṣavantaḥ ||

'A certain sacrificer born in the village named Śivapura composed (*vyadhita*) a treatise called *Karaṇapaddhati*. Let the good learned people after having studied this mathematical text thoroughly be satisfied from what has been said here.'

In his judgement, the words *gaṇitametat samyak* expressed 1765653 days of the Kaliyuga, that is, A.D. 1733 as the date of the text.[11] Presumably Whish used the *Kaṭapayadi* system of writing in arriving at the *Kali chronogram*, a system much in use in South India in the medieval times.

The essential features of the *Kaṭapayadi* alphabetical system of number writing are as follows:

(*a*) A consonant in association with a vowel takes significant values, whereas that without a vowel has no such numerical significance and should be disregarded. These values are:

k	*kh*	*g*	*gh*	*ṅ*	*c*	*ch*	*j*	*jh*	*ñ*
1	2	3	4	5	6	7	8	9	0
ṭ	*ṭha*	*ḍ*	*ḍh*	*ṇ*	*t*	*th*	*d*	*dh*	*n*
1	2	3	4	5	6	7	8	9	0
p	*ph*	*b*	*bh*	*m*					
1	2	3	4	5					
y	*r*	*l*	*v*	*ś*	*ṣ*	*s*	*h*		
1	2	3	4	5	6	7	8		

(*b*) In a conjoint consonant, only the last one denotes a number.

(*c*) In the final reading, the numbers are to be reckoned from right to left.

According to the above system, the words *gaṇitametat samyak* reduce to:

ga —	*ṇi* —	*ta* —	*me* —	*ta* —	*t* —	*sa* —	*mya* —	*k*	= 1765653
3	5	6	5	6	×	7	1	×	

Thus 1765653 days elapsed from the beginning of the *Kali* era till the date in question. Taking 365·25 days as the length of the Indian solar year, and the beginning of the *Kali* era as 17th February, 3102 B.C., the date of the text in the *Gregorian* calendar is obtained as under:

$$\begin{aligned} 1765653 \text{ days} &= (4834 \times 365{\cdot}25 + 34{\cdot}50) \text{ days} \\ &= 4834\ \textit{Kali} \text{ era (approx.)} \\ &= (3102 + 1733)\ \textit{Kali} \text{ era (approx.)} \\ &= \text{A.D. } 1733 \text{ (since } \textit{Kali} \text{ era began at 3102 B.C.)} \end{aligned}$$

By this system, any word can be changed into a chronogram. A careful consideration of the use of the *Kaṭapayadi* systems in verses dealing with astronomical and mathematical subjects reveals that such alphabetical combinations are always readily distinguishable from those signifying accepted word meanings. Moreover, separate words are not used in the manner shown above. The following will illustrate the point:

dīpa	*balaṃ*	*java*	*bāṇā*	*netā*	*kṣāntiḥ*	*kathaṃ*	*mitho*	*dāsāḥ*
18	33	48	53	60	66	71	75	78

pāde	*guhā*	*bhuja*	*madaṃ*	*mudā*	*tadā*	*śukramandajīvaḥ syuḥ* ||
81	83	84	85	85	86	(are the sines of the *manda* (corrections) of the Venus)

(*Dṛggaṇita*,[12] 2, 35)

Whish's selection of words with clear literary meanings is not only arbitrary, but his selection of separate words with a view to obtain a large number is not warranted by the usage of the system. If *gaṇitametat* and *samyak* are converted separately, these give 65653 and 17 which do not lead us anywhere. If the liberty assumed by Whish were possible, one would also be able to convert, say, *ko'pi yajvā kimapi* of the same verse into 1114151 days, that is A.D. 1043, an absurd conclusion as the text would then antidate Bhāskara II (*c.* 1150) whom it quotes at several places. Thus, no word of the concluding verse represents number or date and as such the derivation by Whish cannot be accepted.

Some light on the subject is now available from the studies of Rāja Rāja Varma and Ullur Parameśvara Iyer. Rāja Rāja Varma[13] has remarked that the period of the author of *Karaṇapaddhati* can be placed between the years 550 and 650 of the Malabar era, corresponding roughly to A.D. 1375–1475 and quoted in support of his view a verse from *Gaṇita Sūcikā Grantha* by Govinda Bhaṭṭa. The verse runs thus:

navīna vipine mahīmakhabhujām somayā-
jyudaragaṇa kotraya samabhavacca tenāmunā |
vyalekhi sudṛguttama Karaṇa-paddhati saṃskṛtā
tripañcaśati bhūmita pradhita śaka samvatsare ||

'A *somayājin* versed in astronomy was born in the brāhmin family of Navīna-vipina. By him was composed this refined *Karaṇapaddhati*, the best of the *Dṛg* (system) in 1353 Śaka era (A.D. 1431).'

Ullur Parameśvara Iyer[14] has given the beginning of the seventh century of the Malabar era (fifteenth century A.D.) as the date of the *Karaṇapaddhati*; but no basis for this assertion has been given.

In the circumstances, therefore, a more dependable placement of the time of the work should be attempted from an internal study of the work itself. *Karaṇapaddhati* contains discussions of a number of topics which also reappear in the *Tantrasaṃgraha* of Nīlakaṇṭha Somasuttvan. There is no uncertainty as to Nīlakaṇṭha's date, 1465–1545 A.D., inasmuch as this South Indian astronomer declared himself as the direct pupil* of Parameśvara, the indefatigable commentator and author of astronomical works.[15]

The rule for obtaining the π series already discussed, together with a rule for obtaining an approximate value for the last term, is given in the *Tantrasaṃgraha* as follows[16]:

vyāse vāridhinihate rūpahṛte vyāsasāgarābhihate |
triśarādi-viṣama-saṃkhyā-bhakta-mṛṇaṃ svaṃ pṛthak kramāt kuryāt ||
yatsaṃkhyayātra haraṇe kṛte nivṛttā hṛtistu jāmitayā |
tasyā ūrdhvagatāyāssamasaṃkhyāyā taddalaṃ guṇo'nte syāt |
tadvargaiḥ rūpayuto hāro vyāsābdhighātataḥ prāgvat |
tasyāmāptaṃ svamṛṇe kṛte dhane śodhanañca karaṇīyam ||
sūkṣmaḥ paridhiḥ sā syāt bahukṛtvo haraṇato'tisūkṣmaśca | (ch. 2)

'Quotients obtained by dividing four times of the diameter by the odd integers 3, 5, 7 . . . are alternately subtracted from and added to 4D. The process stops at a certain stage giving rise to a finite sum. Four times of the diameter is to be multiplied by half the even integer and is divided by the square of the even integer increased by unity. This is the last term. The result is the correction to be added to or subtracted from our finite sum (the choice of addition or subtraction depends on the sign of the last term in the

* Nīlakaṇṭha in his *Āryabhaṭīya Bhāṣya* states '*tadeva paramācārya mamāha Parameśvara*' (*Gola*, 48). Parameśvara gives the dates of writing *Dṛggaṇita* and *Goladīpikā* as Śaka 1353 (A.D. 1431) and Śaka 1365 (A.D. 1443) respectively.

sum). The final result is the circumference determined more exactly than by taking a large number of terms.'

The above may be expressed as follows:

$$C = 4D\left[1-\frac{1}{3}+\frac{1}{5}-\ldots\pm\frac{1}{n}\mp\frac{\frac{n+1}{2}}{(n+1)^2+1}\right]$$

where n is odd and large.

As it is a convergent series, the author found it necessary to correct the last quotient.

In another verse of the *Tantrasaṃgraha*,[17] a rule has been given to obtain a still more accurate expression for the last term. The rule runs thus:

> *asmāt sūkṣmataro'nyo vilikhyate kaścanāpi saṃskāraḥ* |
> *ante samasaṃkhyā-dala-vargassaiko guṇassa eva punaḥ* ||
> *yugaguṇito rūpayutassamasaṃkhyādalahato bhavedhāraḥ* | (ch. 2)

'Here another correction is given more precise than the foregoing. The square of half the even integer next (greater than the last odd integer divisor), increased by unity, is a multiplier. This multiplier, multiplied by four, then increased by unity and then multiplied by the even integer already defined, is the divisor.'

Evidently,

$$C = 4D\left[1+\frac{1}{3}-\frac{1}{5}-\ldots\pm\frac{1}{n}\mp\frac{\left(\frac{n+1}{2}\right)^2+1}{\left[\left\{\left(\frac{n+1}{2}\right)^2+1\right\}4+1\right]\left(\frac{n+1}{2}\right)}\right]$$

where n is odd and large.

This shows that the author of the *Tantrasaṃgraha* had definitely the knowledge of a slowly-converging series, of which there is no indication in the *Karaṇapaddhati*. From the much fuller and more refined treatment of these topics in the *Tantrasaṃgraha*, *Karaṇapaddhati* may be suspected to be a work, contemporaneous with, or even antidating, *Tantrasaṃgraha* of Nīlakaṇṭha (A.D 1465–1545).

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References

[1] The last verse of the tenth chapter reads:

'*iti śivapura-nāma grāmjaḥ ko'pi yajvā*'.

[2] *Trivandrum Sanskrit Series* (*T.S.S.*), No. 126.

[3] See *Madras Triennial Catalogue*, Vol. II, Part I. Sans. A., Catalogue No. *1310*.

[4] 'On the Hindu quadrature and the infinite series of the proportion of the circumference to the diameter exhibited in the four śāstras: *Tantrasaṃgraha*, *Yuktibhāṣā*, *Karaṇapaddhati* and *Sadratnamālā*.' *Transaction of the Royal Asiatic Society* (*TRAS*) of Great Britain and Ireland, **3**, Part III, 509–523, 1834.

[5] 'On the Hindu Quadrature of the Circle', *Journal of the Bombay Branch of the Royal Asiatic Society* (*JBBRAS*), **20** (n.s.), 65–82, 1944; 'A neglected chapter of Hindu Mathematics', *Scripta Mathematica*, **15**, 201–209, 1949; 'The Sine and Cosine Power Series', *JASB*, **15**, Science Section, 1–13, 1949; 'On the Hindu proof of Gregori's Series', *Scripta Mathematica*.

[6] 'The development of Mathematical Series in India after Bhāskara II', *Bulletin of the National Institute of Sciences of India* (*BNISI*), **21**, 320–43, 1963.

[7] *See* M. Cantor's *Vorlesungen Über Geschichte der Mathematik*, 2nd ed., **3** (c/o *JASB*, **15**, Part II, 1, 1949).

[8] *TRAS*, of Great Britain and Ireland, 3, Part III, 513, 516, 519.

[9] *BNISI*, **21**, 320–43, 1963; *JBBRAS*, **20** (n.s.), 66–69, 1944.

[10] *Chambers's Encyclopaedia*, **10**, 227, 1935; Cajori, F., *A History of Mathematics*, 2nd revised and enlarged ed., 206.

[11] *TRAS*, of Great Britain and Ireland, **3**, Part III, 522, 1834.

[12] *Dṛggaṇita* of Parameśvara, ed. by K. V. Sarma, *Vishveshavanand Vedic Research Institute*, Hoshiarpur, 1963.

[13] Rāja Rāja Varma, *History of Sanskrit Literature in Kerala*, **1**, 529.

[14] Ullur, Parameśvara Iyer, *Kerala Sahitya Caritra*, **2**, 564.

[15] See *A Bibliography of Sanskrit Works on Astronomy and Mathematics* by S. N. Sen, A. K. Bag and S. R. Sarma, 167 (under PARAMEŚVARA).

[16] MSS. of *Trippunitura Sanskrit College Library and the Adyar Library*, ch. 2 (c/o *JBBRAS*, **20** (n.s.), 77, 1944).

[17] See *JBBRAS*, **20** (n.s.), 81, 1944.